

# EQUIMULTIPLICITY IN HILBERT-KUNZ THEORY

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*Dedicated to Craig Huneke on the Occasion of his 65th Birthday.*

ABSTRACT. This paper develops a theory of equimultiplicity for Hilbert-Kunz multiplicity and applies it to study the behavior of Hilbert-Kunz multiplicity on the Brenner-Monsky hypersurface. A number of applications follows, in particular we show that Hilbert-Kunz multiplicity attains infinitely many values and that equimultiple strata may not be locally closed.

## 1. INTRODUCTION

Hilbert-Samuel multiplicity is a classical invariant of a local ring that generalizes the notion of the multiplicity of a curve at a point. The multiplicity may be regarded as a measure of singularity, where the lowest possible value, 1, corresponds to a smooth point. In study of singularities, we are naturally led to study equimultiple points, i.e., a point such that the Hilbert-Samuel multiplicity is constant on the subvariety defined as the closure of the point. For example, it can be considered as the weakest form of equisingularity, where we would say that two points are equally singular if the multiplicities are equal. This notion has been studied extensively, partially due to its appearance in Hironaka's work on the resolution of singularities in characteristic zero.

In 1983 Monsky defined a new version of multiplicity, specific for positive characteristic. It was called Hilbert-Kunz multiplicity, in honor of Ernst Kunz who in 1969 initiated the study of positive characteristic numerical invariants ([14, 15]). If we use  $\lambda$  to denote the length of an artinian module, the Hilbert-Kunz multiplicity of a local ring  $(R, \mathfrak{m})$  of characteristic  $p$  can be defined by the limit

$$e_{\text{HK}}(R) = \lim_{n \rightarrow \infty} \frac{1}{p^{n \dim R}} \lambda(R/\mathfrak{m}^{[p^n]}),$$

where we denoted the ideal of  $p^n$ -powers,  $(x^{p^n} \mid x \in \mathfrak{m})$ , by  $\mathfrak{m}^{[p^n]}$ . This invariant is still largely mysterious, and there is a lot to understand. The purpose of this work is to build an equimultiplicity theory for Hilbert-Kunz multiplicity and to see what properties of the usual multiplicity are preserved.

As a tool to study singularities, Hilbert-Kunz multiplicity shares some properties with the classical multiplicity, but often has a more complicated behavior. While the Hilbert-Samuel multiplicity of a local ring is always an integer, the Hilbert-Kunz multiplicity is not. However, this also allows us to think about Hilbert-Kunz multiplicity as a finer invariant. Just like the usual multiplicity, it detects regular rings (Watanabe and Yoshida, [30]); but it is also meaningful to consider rings of Hilbert-Kunz multiplicity sufficiently close to 1, and expect that their singularities should get better. Blickle and Enescu ([6] and its improvement [1] by Aberbach and Enescu) showed that rings with sufficiently small Hilbert-Kunz multiplicity are Gorenstein and F-regular.

These results show that Hilbert-Kunz multiplicity is a measure of singularity, and the next natural step is to study its geometric properties, with a view towards a possible use for the resolution of singularities. For a possible list of desired properties let me quote Lipman's featured review

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([4, pp. B28-B33]) of the celebrated paper of Bierstone and Milman ([5]): “In the early 1980’s, Abhyankar announced a “canonical desingularization” procedure ([3]). His fundamental idea was, very roughly, to construct a desingularization as a succession of blowups, specifying at every point of each variety appearing in the process a function, the “resolution invariant,” which is locally invariant in the sense of compatibility with open immersions, which depends on the preceding history of the process, which takes a finite number of values in a totally ordered set, and whose maximum value locus is a smooth subvariety, upon blowing up which one lowers the said maximum value, thereby “improving the worst singularities”. Eventually, by finiteness, the resolution invariant can no longer decrease – so it is constant everywhere and then the original variety has been transformed into a smooth one.”

Previously, the author showed that Hilbert-Kunz multiplicity is upper semi-continuous ([27]); essentially this ensures that the maximum value locus is a subvariety. Following the aforementioned review of Lipman and the resolution of Bierstone and Milman ([5]), we may ask whether Hilbert-Kunz multiplicity satisfies the stronger condition of Zariski-semicontinuity: thus Hilbert-Kunz multiplicity on a fixed variety should attain only finitely many values. As observed in [5, Lemma 3.10], Zariski-semicontinuity can be restated as openness for all  $a$  of the sets

$$X_{\leq a} = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid e_{\text{HK}}(\mathfrak{p}) \leq a\}.$$

After using Nagata’s criterion for openness, this will require us to prove that for some  $s \notin \mathfrak{p}$  we can make the Hilbert-Kunz multiplicity to be constant on  $V(\mathfrak{p}) \cap D(s)$ , or, as it fits to say, make  $\mathfrak{p}$  *equimultiple* by inverting an element. In order to do so, we first need to understand what ideals will satisfy this version of equimultiplicity and the goal of this paper is to build such understanding.

**1.1. Overview of the results.** Before proceeding to our results, let us quickly review the classical picture. The analytic spread of an ideal  $I$ ,  $\ell(I)$ , is defined as the Krull dimension of the fiber cone

$$F(I) = R/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \cdots.$$

The analytic spread plays an important role in the classical Hilbert-Samuel theory, because it determines the size of a minimal reduction, i.e., the number of general elements in  $I$  needed to generate  $I$  up to integral closure. Let us denote the set of all prime ideals  $P \supseteq I$  such that  $\dim R/P = \dim R/I$  by  $\operatorname{Minh}(I)$ , and recall the classical characterization of equimultiple ideals ([24, 17]).

**Theorem 1.1.** *Let  $(R, \mathfrak{m}, k)$  be a formally equidimensional local ring. For an ideal  $I$  in  $R$  the following conditions are equivalent:*

- (a)  $\ell(I) = \operatorname{ht}(I)$ ,
- (b) for every (equivalently, some) parameter ideal  $J$  modulo  $I$

$$e(I + J) = \sum_{P \in \operatorname{Minh}(I)} e(J, R/P) e(I, R_P),$$

- (c) if  $k$  is infinite, there is a system of parameter  $J = (x_1, \dots, x_r)$  modulo  $I$  which is a part of a system of parameters in  $R$ , and such that
  - if  $r < \dim R$  then  $\sum_{P \in \operatorname{Minh}(I)} e(J, R/P) e(IR_P) = e(IR/J)$
  - if  $r = \dim R$  then  $\sum_{P \in \operatorname{Minh}(I)} e(J, R/P) e(IR_P) = e(J)$ ,
- (d) for every (equivalently, some) system of parameters  $(x_1, \dots, x_r)$  modulo  $I$ , all  $0 \leq i < r$ , and all  $n$

$$x_{i+1} \text{ is regular modulo } \overline{(I, x_1, \dots, x_i)^n}.$$

An ideal that satisfies any of these equivalent condition is called *equimultiple*. This name originates from the special case when  $I = \mathfrak{p}$  is a prime ideal such that  $R/\mathfrak{p}$  is regular. In this case the condition (b) shows that  $e(\mathfrak{m}) = e(\mathfrak{p})$ .

In this paper we develop a theory of *Hilbert-Kunz equimultiple* ideals. Again, if  $I = \mathfrak{p}$  is prime and  $R/\mathfrak{p}$  is regular, then  $\mathfrak{p}$  is Hilbert-Kunz equimultiple if and only if  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$ . The obtained characterization is very similar: we replace powers and integral closure with Frobenius powers and tight closure. The most important difference is that an analogue of part (a) is missing because we are not able to find a satisfactory replacement for analytic spread. Our findings are collected in the following theorem (Theorem 5.17, Corollaries 5.26 and 5.34).

**Theorem 1.2.** *Let  $(R, \mathfrak{m})$  be a local ring of positive characteristic  $p > 0$ . Furthermore suppose that  $R$  is either formally unmixed with a locally stable test element  $c$  or is a excellent and equidimensional (e.g., a complete domain). For an ideal  $I$  the following conditions are equivalent:*

(b) *for every (equivalently, some) parameter ideal  $J$  modulo  $I$*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P),$$

(c) *there is a system of parameter  $J = (x_1, \dots, x_r)$  modulo  $I$  which is a part of a system of parameters in  $R$ , and such that*

- *if  $r < \dim R$  then  $\sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(IR_P) = e_{\text{HK}}(IR/J)$*
- *if  $r = \dim R$  then  $\sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(IR_P) = e_{\text{HK}}(J)$ ,*

(d) *for every (equivalently, some) system of parameters  $(x_1, \dots, x_r)$  modulo  $I$ , all  $0 \leq i < r$ , and all  $n$*

$$x_{i+1} \text{ is regular modulo } ((I, x_1, \dots, x_i)^{[q]})^*.$$

The main consequence of this characterization is a surprising negative answer (Corollary 6.10): Hilbert-Kunz multiplicity is not Zariski-semicontinuous in sense of [5].

**Corollary 1.3.** *Let  $R = F[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2)$ , where  $F$  is the algebraic closure of  $\mathbb{Z}/2\mathbb{Z}$ . Then the set*

$$X_{\leq 3} = \{\mathfrak{p} \in \text{Spec}(R) \mid e_{\text{HK}}(\mathfrak{p}) \leq 3\}$$

*is not open.*

This readily implies that Hilbert-Kunz multiplicity attains infinitely many values (Corollary 6.11) on the Brenner-Monsky hypersurface  $z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2 = 0$ . This hypersurface was previously used by Brenner and Monsky to provide a counter-example to the notorious localization problem in tight closure theory ([8]).

In order to understand the Brenner-Monsky counter-example better and compare it to the results of [12], Dinh have studied the associated primes of the Frobenius powers of the prime ideal  $P = (x, y, z)$  defining the singular locus of the aforementioned hypersurface. Using the calculations that Monsky made to obtain Theorem 6.8, Dinh ([9]) proved that  $\bigcup_q \text{Ass}(P^{[q]})$  is infinite. However, he was only able to show that the maximal ideals corresponding to the irreducible factors of  $1 + t + t^2 + \dots + t^q$  appear as associated primes, while our methods immediately give all associated primes of the Frobenius powers and their tight closures.

**Proposition 1.4.** *In the Brenner-Monsky example,*

$$\bigcup_q \text{Ass}(P^{[q]})^* = \bigcup_q \text{Ass}(P^{[q]}) = \text{Spec}(R/P).$$

*In particular, the set is infinite.*

Our applications demonstrate that in Hilbert-Kunz theory equimultiplicity is a far more restrictive condition than equimultiplicity in Hilbert-Samuel theory. This fits well with our understanding

of Hilbert-Kunz multiplicity as a finer invariant; when there are many more possible values it is less likely for the values at two points to coincide.

The presented applications were obtained in the Brenner-Monsky hypersurface because we used on Monsky's computations in [22]. Potentially the theory can be applied to any other example where the Hilbert-Kunz multiplicity was computed in a family (e.g., [21]); we chose the simplest and the most understood family.

It is also worth to highlight a very surprising corollary (Corollary 5.19) obtained by combining Theorem 1.2 with Proposition 3.1 that gives a criterion for equality of the entire Hilbert-Kunz functions of  $\mathfrak{p}$  and  $\mathfrak{m}$ .

**Corollary 1.5.** *Let  $(R, \mathfrak{m})$  be a weakly  $F$ -regular excellent local domain and let  $\mathfrak{p}$  be a prime ideal such that  $R/\mathfrak{p}$  is regular. Then  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$  if and only if the entire Hilbert-Kunz functions (not just the limits!) of  $R$  and  $R_{\mathfrak{p}}$  coincide.*

Last, I want to remark that the presented theory has recently found an application in Linquan Ma's breakthrough on Lech's conjecture ([18]). As a key ingredient of his proof, Ma showed that perfect ideals are Hilbert-Kunz equimultiple.

**1.2. The structure of the paper.** The absence of reductions and analytic spread required us to develop new tools. The uniform convergence methods provide a crucial tool for our proof, Corollary 4.11, and we use Tucker's ideas from [29] to develop Section 4. Corollary 4.11 is interesting on its own right and will be useful in a variety of situations (such as Ma's recent work in [18]).

We use this tool in Section 5, where we build the bulk of the Hilbert-Kunz equimultiplicity theory, resulting in Theorem 5.17 and its corollaries. Perhaps the most challenging part of the proof is an analogue of the equivalence between (b) to (d) of Theorem 1.1, which we achieve in Theorems 5.9, 5.11. Later in the section we study the general behavior of Hilbert-Kunz equimultiplicity under ring operations in order to relax some of the assumptions of the aforementioned theorems. We also investigate the equivalence between "some" and "every" in that analogue of part (d) under milder assumptions (Corollary 5.27). In the end of the section, we study the behavior of Hilbert-Kunz equimultiplicity under localization and specialization: Proposition 5.28 shows that the property is preserved under localization, and Corollary 5.29 provides a Bertini-type result. These results are used to establish, in Corollary 5.34, the condition (c) of Theorem 1.2.

In Section 6, we will use the machinery we have constructed to study the Brenner-Monsky hypersurface and obtain a number of interesting results. This hypersurface was previously used by Brenner and Monsky to provide a counter-example to the notorious localization problem in tight closure theory ([8]).

As the first step, in Corollary 6.6 we will establish equivalence between equimultiplicity of prime ideal  $\mathfrak{p}$  of dimension one and  $\mathfrak{p}$ -primary property of the tight closures  $(\mathfrak{p}^{[q]})^*$ . Then we use the previously mentioned results on specialization (Corollaries 5.31, 5.32) in Proposition 6.9, where we derive from Monsky's computations in [22] that any point on the curve defining the singular locus has Hilbert-Kunz multiplicity greater than the generic point. This implies that the Hilbert-Kunz multiplicity attains infinitely many values on the curve. Hence, the stratification by Hilbert-Kunz equimultiple strata need not be finite (Corollary 6.11). Also, in Corollary 6.10, we will show that an individual stratum need not be locally closed and the set  $X_{\leq a}$  need not be open.

As a second application of our methods, in Proposition 6.12 we easily compute the set of all associated primes of the Frobenius powers (or tight closures of the Frobenius powers) for the defining ideal of the previously mentioned curve. This improves a result of Dinh ([9]) who showed that the set is infinite.

In the end of the paper, we give a list of open problems further in the direction of completed work.

## 2. PRELIMINARIES

In this paper we study commutative Noetherian rings, typically of characteristic  $p > 0$ . For convenience, we use notation  $q = p^e$  where  $e \in \mathbb{Z}$  may vary. For an ideal  $I$  of  $R$  let  $I^{[q]}$  be the ideal generated by  $q$ th powers of the elements of  $I$ .

**2.1. Tight closure.** Let us briefly review some results from tight closure theory that will be needed for our work. For proofs and a more detailed exposition we refer a curious reader to [10].

**Definition 2.1.** Let  $R$  be a ring and let  $I$  be an ideal of  $R$ . The tight closure  $I^*$  of  $I$  consists of all elements  $x$  of  $R$  such that there exists a fixed element  $c$  such that  $c$  is not contained in any minimal prime of  $R$  and

$$cx^q \in I^{[q]}$$

for all sufficiently large  $q$ .

The following properties can be found in [10, Proposition (4.1)].

**Proposition 2.2.** *Let  $R$  be a ring and let  $I, J$  be ideals of  $R$ .*

- (1)  $I^*$  is an ideal.
- (2)  $(I^*)^* = I^*$ ,
- (3) If  $I \subseteq J$  then  $I^* \subseteq J^*$ .
- (4) For all  $q$   $(I^*)^{[q]} \subseteq (I^{[q]})^*$ .
- (5)  $x \in I^*$  if and only if the image of  $x$  is in  $(IR/P)^*$  for all minimal primes  $P$ .
- (6)  $I^*$  is the preimage of  $(IR_{\text{red}})^*$  in  $R$ .

**Definition 2.3.** Let  $R$  be a Noetherian ring of characteristic  $p > 0$ . We say that an element  $c$  is a test element if  $c$  is not contained in any minimal prime and for every ideal  $I$  and element  $x \in R$   $x \in I^*$  if and only if  $cx^q \in I^{[q]}$  for all  $q$ .

Test elements make tight closure a very powerful tool and have had a great use. After a tremendous amount of work in [11, Theorem 6.1], Hochster and Huneke obtained the following existence theorem.

**Theorem 2.4.** *Let  $R$  be reduced and either  $F$ -finite or of essentially finite type over an excellent local ring. Then  $R$  has a test element.*

Test elements provide us with a very useful lemma.

**Lemma 2.5.** *Let  $(R, \mathfrak{m})$  be a local ring of positive characteristic with a test element  $c$ . If  $I$  and  $J$  are two proper ideals then*

$$I^* = \cap_q (I, J^{[q]})^*.$$

*Proof.* This easily follows from the definition: if  $x$  belongs to the intersection, then

$$cx^{q'} \in \cap_n (I, J^n)^{[q']} \text{ (or } \cap_q (I, J^{[q]})^{[q']})$$

and we are done by Krull's intersection theorem. □

**2.2. Tight closure and Hilbert-Kunz multiplicities.** For our work we will need to extend the definition given in Introduction. Its existence was still proved by Monsky ([20]).

**Definition 2.6.** Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$ ,  $I$  be an  $\mathfrak{m}$ -primary ideal, and  $M$  be a finite  $R$ -module. The limit

$$e_{\text{HK}}(I, M) := \lim_{q \rightarrow \infty} \frac{1}{q^{\dim R}} \lambda(M/I^{[q]}M),$$

is called the Hilbert-Kunz multiplicity of  $M$  with respect to  $I$ .

If  $I = \mathfrak{m}$ , we will call it the *Hilbert-Kunz multiplicity* of  $M$  and denote  $e_{\text{HK}}(M)$ . For a prime ideal  $\mathfrak{p}$ , it is convenient to denote  $e_{\text{HK}}(\mathfrak{p}R_{\mathfrak{p}})$  by  $e_{\text{HK}}(\mathfrak{p})$ . This naturally makes Hilbert-Kunz multiplicity to be a function on the spectrum and the goal of this paper is to understand when  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$ .

Now we discuss a very useful connection between Hilbert-Kunz multiplicity and tight closure. For two ideals  $I \subset J$ , it is easy to see that  $e_{\text{HK}}(I) \geq e_{\text{HK}}(J)$ , but an equality may hold despite that the two ideals are distinct. The following theorem, due to Hochster and Huneke ([10, Theorem 8.17]), describes when does it happen.

Recall that a local ring is formally unmixed if  $\text{Ass}(0\widehat{R}) = \text{Min}(0\widehat{R})$ , or, in other words,  $\dim \widehat{R}/P = \dim R$  for every associated prime of  $\widehat{R}$ . For example, a complete domain is formally unmixed.

**Theorem 2.7.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring and  $I \subseteq J$  are ideals. Then  $J \subseteq I^*$  if and only if  $e_{\text{HK}}(I) = e_{\text{HK}}(J)$ .*

The following lemma shows that the Hilbert-Kunz multiplicity of  $I$  can be computed using the filtration  $(I^{[q]})^*$ ; it can be thought of as a generalization of the “only if” direction. We are interested in this filtration, since it is often more useful than the usual filtration  $I^{[q]}$ .

*Remark 2.8.* If  $R$  has a test element  $c$ , then, by definition,  $c\sqrt{0} = c0^* = 0$ . Since  $c$  does not belong to any minimal prime, it follows that  $R_P$  is reduced for any minimal prime  $P$  of  $R$ . Therefore the dimension of the nilradical of  $R$  is less than  $\dim R$ . Thus  $e_{\text{HK}}(I, M) = e_{\text{HK}}(I, R_{\text{red}} \otimes_R M)$  for any  $\mathfrak{m}$ -primary ideal  $I$  and module  $M$ .

**Lemma 2.9.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$ ,  $I$  be an  $\mathfrak{m}$ -primary ideal, and  $M$  a finitely generated  $R$ -module. If  $R$  has a test element  $c$ , then*

$$\lim_{q \rightarrow \infty} \frac{1}{q^d} \lambda(M/(I^{[q]})^* M) = e_{\text{HK}}(I, R_{\text{red}} \otimes_R M) = e_{\text{HK}}(I, M).$$

*Proof.* First, by its definition and (6) of Proposition 2.2,  $c$  is still a test element in  $R_{\text{red}}$ . Also Proposition 2.2 shows that  $R/(I^{[q]})^* \cong R_{\text{red}}/(I^{[q]}R_{\text{red}})^*$ , and, using the previous remark, we assume that  $R$  is reduced.

Now, consider an exact sequence

$$R \xrightarrow{c} R \rightarrow R/(c) \rightarrow 0.$$

Since  $c(I^{[q]})^* \subseteq I^{[q]}$ , we obtain that the sequence

$$R/(I^{[q]})^* \otimes_R M \xrightarrow{c} R/I^{[q]} \otimes_R M \rightarrow R/(c, I^{[q]}) \otimes_R M \rightarrow 0.$$

is still exact. Together with inclusion  $I^{[q]} \subseteq (I^{[q]})^*$  this shows that

$$\lambda(M/(I^{[q]})^* M) \leq \lambda(M/I^{[q]} M) \leq \lambda(M/(I^{[q]})^* M) + \lambda(M/(c, I^{[q]}) M).$$

Since  $c$  is not contained in any minimal prime,  $\dim M/cM \leq \dim R/(c) < \dim R$ . Therefore  $e_{\text{HK}}(I, M) = \lim_{q \rightarrow \infty} \frac{1}{q^d} \lambda(M/(I^{[q]})^* M)$ .  $\square$

We will need the following corollary to deal with localization of tight closure. Unfortunately, tight closure does not commute with localization, but there is still an inclusion  $I^*R_{\mathfrak{p}} \subseteq (IR_{\mathfrak{p}})^*$ , where the first closure is taken in  $R$  and the second is in  $R_{\mathfrak{p}}$ . Thus, the corollary allows us to compute the Hilbert-Kunz multiplicity of  $R_{\mathfrak{p}}$  by taking the filtration  $(\mathfrak{p}^{[q]})^*R_{\mathfrak{p}}$ .

**Corollary 2.10.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  with a test element  $c$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal and  $I_q$  be a sequence of ideals such that  $I^{[q]} \subseteq I_q \subseteq (I^{[q]})^*$ . Then*

$$\lim_{q \rightarrow \infty} \frac{1}{q^d} \lambda(R/I_q) = e_{\text{HK}}(I).$$

*Proof.* The claim follows from Lemma 2.9, since the inclusions  $I^{[q]} \subseteq I_q \subseteq (I^{[q]})^*$  give that

$$e_{\text{HK}}(I) \geq \lim_{q \rightarrow \infty} \frac{1}{q^d} \lambda(R/I_q) \geq \lim_{q \rightarrow \infty} \frac{1}{q^d} \lambda(R/(I^{[q]})^*) = e_{\text{HK}}(I).$$

□

More importantly, there is a partial converse to this inequality improving further Theorem 2.7. It will be very useful later, as it provides us a way to detect when a filtration is in tight closure.

**Lemma 2.11.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring of characteristic  $p > 0$  and  $I$  be an  $\mathfrak{m}$ -primary ideal. If  $I_q$  is a sequence of ideals such that  $I^{[q]} \subseteq I_q$ ,  $I_q^{[q']} \subseteq I_{qq'}$  for all  $q, q'$ , and*

$$\lim_{q \rightarrow \infty} \frac{\lambda(R/I_q)}{q^d} = e_{\text{HK}}(I),$$

*then  $I_q \subseteq (I^{[q]})^*$  for all  $q$ .*

*Proof.* By the assumptions on  $I_q$ ,

$$I^{[qq']} \subseteq I_q^{[q']} \subseteq I_{qq'}.$$

Therefore,

$$q^d e_{\text{HK}}(I) = e_{\text{HK}}(I^{[q]}) \geq e_{\text{HK}}(I_q) \geq \lim_{q \rightarrow \infty} \frac{\lambda(R/I_{qq'})}{(q')^d} = q^d e_{\text{HK}}(I),$$

so  $e_{\text{HK}}(I^{[q]}) = e_{\text{HK}}(I_q)$ . Hence by Theorem 2.7,  $I_q \subseteq (I^{[q]})^*$ . □

The following lemma helps us to understand what it means for an element to be regular modulo tight closures of consecutive powers. Recall that for an ideal  $I$  and element  $x \notin I$  we denote  $I : x^\infty = \bigcup_n I : x^n$ .

**Lemma 2.12.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$ ,  $I$  an ideal, and  $x$  an element. Suppose that  $R$  has a test element  $c$ , then the following are equivalent:*

- (a)  *$x$  is not a zero divisor modulo  $(I^{[q]})^*$  for any  $q$ ,*
- (b)  *$I^{[q]} : x^\infty \subseteq (I^{[q]})^*$  for all  $q$ ,*
- (c) *for all  $q$  there are ideals  $I_q$  such that  $x$  is not a zero divisor modulo  $I_q$  and  $I^{[q]} \subseteq I_q \subseteq (I^{[q]})^*$ .*

*Proof.* (a)  $\Rightarrow$  (b) since  $I^{[q]} : x^\infty \subseteq (I^{[q]})^* : x^\infty$ . (b)  $\Rightarrow$  (c) trivially. For the last implication, we observe that if  $ax \in (I^{[q]})^*$  for some  $q$ , then  $ca^{q'}x^{q'} \in I^{[qq']}$  for all  $q' > 0$ . But, since  $x^{q'}$  is not a zero divisor modulo  $I_{qq'}$ ,

$$ca^{q'} \in I_{qq'} \subseteq (I^{[qq']})^*.$$

Since  $R$  has a test element, these equations imply that  $a \in (I^{[q]})^*$ . □

**Corollary 2.13.** *Let  $(R, \mathfrak{m})$  be a local ring of positive characteristic and  $\mathfrak{p}$  be a prime ideal. If  $R$  has a test element  $c$ , the following are equivalent:*

- (a)  *$(\mathfrak{p}^{[q]})^*$  is  $\mathfrak{p}$ -primary for any  $q$ ,*
- (b)  *$\mathfrak{p}^{[q]}R_{\mathfrak{p}} \cap R \subseteq (\mathfrak{p}^{[q]})^*$  for all  $q$ ,*
- (c) *for all  $q$  there exist  $\mathfrak{p}$ -primary ideals  $I_q$  such that  $\mathfrak{p}^{[q]} \subseteq I_q \subseteq (\mathfrak{p}^{[q]})^*$ .*

*Proof.* Clearly,  $I_q = \mathfrak{p}^{[q]}R_{\mathfrak{p}} \cap R$  yields (b)  $\Rightarrow$  (c).

For an ideal  $I$  such that  $\sqrt{I} = \mathfrak{p}$ , we can characterize its  $\mathfrak{p}$ -primary part as the smallest among the ideals containing  $I$  and such that any element  $x \notin \mathfrak{p}$  is not a zero divisor modulo that ideal. Thus, for (c)  $\Rightarrow$  (a), we note that  $\mathfrak{p}^{[q]} : x^\infty \subseteq I_q$  and use the lemma above.

For the last implication, we just note that  $\mathfrak{p}^{[q]}R_{\mathfrak{p}} \cap R \subseteq (\mathfrak{p}^{[q]})^*R_{\mathfrak{p}} \cap R$ . □

**2.3. Basic properties of multiplicities.** Let us overview some properties of multiplicities and Hilbert-Kunz multiplicities that will be used through the text.

First, both multiplicities satisfy the associativity formula.

**Lemma 2.14** (Associativity formula). *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d$  and  $I$  an arbitrary  $\mathfrak{m}$ -primary ideal. Then*

$$e(I) = \sum_{P \in \text{Minh } R} e(IR/P) \lambda_{R_P}(R_P),$$

where the sum is taken over all primes  $P$ , such that  $\dim R/P = \dim R$ .

Similarly, if  $R$  has positive characteristic,

$$e_{\text{HK}}(I) = \sum_{P \in \text{Minh } R} e_{\text{HK}}(IR/P) \lambda_{R_P}(R_P),$$

It is well-known that the multiplicity of a regular sequence can be computed as a colength.

**Proposition 2.15.** *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d$ ,  $x_1, \dots, x_r$  a system of parameters, and  $I$  an arbitrary  $\mathfrak{m}$ -primary ideal. Then  $\lambda(R/(x_1, \dots, x_r)) \geq e((x_1, \dots, x_r))$ .*

Moreover,  $\lambda(R/(x_1, \dots, x_r)) = e((x_1, \dots, x_r))$  if and only if  $x_1, \dots, x_r$  form a regular sequence.

It was shown by Lech ([16]) that for a parameter ideal  $(x_1, \dots, x_d)$ , its multiplicity can be computed by a formula

$$e((x_1, \dots, x_r)) = \lim_{\min(n_i) \rightarrow \infty} \frac{\lambda(R/(x_1^{n_1}, \dots, x_d^{n_d}))}{n_1 \cdots n_d}.$$

This easily implies two very useful corollaries.

**Corollary 2.16.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  and let  $x_1, \dots, x_d$  be a system of parameters. Then*

$$e_{\text{HK}}((x_1, \dots, x_d)) = e((x_1, \dots, x_d)).$$

**Corollary 2.17.** *Let  $(R, \mathfrak{m})$  be a local ring and let  $x_1, \dots, x_d$  be a system of parameters. Then for any vector  $(n_1, \dots, n_d) \in \mathbb{N}^d$*

$$e((x_1^{n_1}, \dots, x_d^{n_d})) = n_1 \cdots n_d e((x_1, \dots, x_d)).$$

Lech also used this observation to obtain the following variation of the associativity formula for parameter ideals ([16, Theorem 1], [23, Theorem 24.7]).

**Proposition 2.18.** *Let  $(R, \mathfrak{m})$  be a local ring and let  $x_1, \dots, x_d$  be a system of parameters. Then for any  $0 \leq i \leq d$*

$$e((x_1, \dots, x_d)) = \sum_{P \in \text{Minh}((x_1, \dots, x_i))} e((x_{i+1}, \dots, x_d), R/P) e((x_1, \dots, x_i)R_P).$$

### 3. EQUIMULTIPLICITY FOR HILBERT-KUNZ FUNCTIONS

First recall that the sequence

$$q \rightarrow \lambda_q(\mathfrak{m}) := \frac{1}{q^{\dim R}} \lambda(R/\mathfrak{m}^{[q]}R),$$

is called the (normalized) Hilbert-Kunz function of a local ring  $(R, \mathfrak{m})$ . As a first step, let us characterize for what prime ideals  $\mathfrak{p}$  the entire Hilbert-Kunz functions of  $R_{\mathfrak{p}}$  and  $R$  coincide. The presented argument is based on the proof of [13, Theorem 3.3].



**Proposition 3.1.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  and  $\mathfrak{p}$  be a prime ideal of  $R$  such that  $R/\mathfrak{p}$  is regular and  $\text{ht } \mathfrak{p} + \dim R/\mathfrak{p} = \dim R$ . Then, for a fixed  $q$ ,  $\lambda_q(\mathfrak{m}) = \lambda_q(\mathfrak{p})$  if and only if  $R/\mathfrak{p}^{[q]}$  is Cohen-Macaulay.*

*Therefore the (normalized) Hilbert-Kunz functions of  $\mathfrak{m}$  and  $\mathfrak{p}$  coincide for all  $q$  if and only if  $R/\mathfrak{p}^{[q]}$  are Cohen-Macaulay for all  $q$ .*

*Proof.* Let  $x_1, \dots, x_m$  be a minimal system of generators of  $\mathfrak{m}$  modulo  $\mathfrak{p}$ . By the associativity formula and Corollary 2.17,

$$e((x_1, \dots, x_m)^{[q]}, R/\mathfrak{p}^{[q]}) = e((x_1, \dots, x_m)^{[q]}, R/\mathfrak{p}) \lambda_{R/\mathfrak{p}}(R/\mathfrak{p}^{[q]} R/\mathfrak{p}) = q^m \lambda_{R/\mathfrak{p}}(R/\mathfrak{p}^{[q]} R/\mathfrak{p}).$$

Since  $q^m \lambda_{R/\mathfrak{p}}(R/\mathfrak{p}^{[q]} R/\mathfrak{p}) = q^{\dim R} \lambda_q(\mathfrak{p})$ , we obtain by Proposition 2.15 that

$$q^{\dim R} \lambda_q(\mathfrak{m}) = \lambda(R/(\mathfrak{p}^{[q]}, x_1^q, \dots, x_m^q)) \geq e((x_1, \dots, x_m)^{[q]}, R/\mathfrak{p}^{[q]}) = q^{\dim R} \lambda_q(\mathfrak{p}).$$

Thus,  $\lambda_q(\mathfrak{m}) = \lambda_q(\mathfrak{p})$  if and only if  $\lambda(R/(\mathfrak{p}^{[q]}, x_1^q, \dots, x_m^q)) = e((x_1, \dots, x_m)^{[q]}, R/\mathfrak{p}^{[q]})$ . However, by Proposition 2.15, the latter holds if and only if  $R/\mathfrak{p}^{[q]}$  is Cohen-Macaulay.  $\square$

While the characterization is simple and natural, it is not clear whether we could force this condition by inverting an element. For a fixed  $q$  this condition holds on an open set  $D(s_q)$  because the Cohen-Macaulay locus is open in an excellent ring. However, we will need to intersect infinitely many open sets  $D(s_q)$  to force it for all  $q$ .

This is why we need to go further and try to characterize equality of Hilbert-Kunz multiplicities. It will be much harder to achieve, but having a better control over Hilbert-Kunz multiplicity, we will learn in Proposition 6.9 that, in fact, the conditions of Proposition 3.1 cannot be forced for all  $q$  by inverting a single element.

Last, we remark that this characterization can be used to give a different proof of the following classical theorem that was first proved by Kunz in [15, Corollary 3.4] (and another simple proof is due to Shepherd-Barron in [25]).

**Theorem 3.2.** *If  $R$  is an excellent locally equidimensional ring, then for any fixed  $q$  the  $q$ th Hilbert-Kunz function  $\lambda_q$  is locally constant on  $\text{Spec}(R)$ .*

#### 4. A UNIFORM CONVERGENCE RESULT

Before proceeding to technicalities, let us sketch the ideas of the proof. Over a local ring  $(R, \mathfrak{m})$  we are going to prove uniform convergence (with respect to  $q$ ) of the bisequence

$$\frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}}$$

where  $I$  is an ideal,  $M$  is a finitely generated module, and  $J$  is an  $\mathfrak{m}$ -primary ideal. This allows us to interchange the limits (with respect to  $q$  and  $q'$ ) of the bisequence.<sup>1</sup>

Uniform convergence will be established by showing that the sequence is Cauchy with an appropriate estimate; to do the bookkeeping of estimates we follow Tucker's treatment in [29]. Tucker's proof can be viewed as a careful adaptation of the original proof by Monsky ([20]).

We start with an upper bound for a function that we study.

**Lemma 4.1.** *Let  $(R, \mathfrak{m})$  be a local ring and  $I$  be an ideal. Then there exists a constant  $C$  such that*

$$\lambda(R/(I^{[q]} + \mathfrak{m}^{[qq']})) \leq C q'^{\dim R/I} q^{\dim R}$$

*for all  $q, q'$ .*

<sup>1</sup>Neil Epstein informed me that a more general change of limits formula was independently obtained in his joint work with Yongwei Yao.

*Proof.* Let  $x_1, \dots, x_h$  be elements of  $R$  such that their images form a system of parameters in  $R/I$ . Then

$$\lambda(R/(I^{[q]} + \mathfrak{m}^{[qq']})) \leq \lambda(R/(I^{[q]} + (x_1, \dots, x_h)^{[qq']})) = \lambda(R/(I^{[q]} + (x_1^{qq'}, \dots, x_h^{qq'}))).$$

Filtering by the powers of  $x_i^q$ , we get that

$$\lambda(R/(I^{[q]} + (x_1^{qq'}, \dots, x_h^{qq'}))) \leq (q')^h \lambda(R/(I, x_1, \dots, x_h)^{[q]}).$$

Since  $(I, x_1, \dots, x_h)$  is an  $\mathfrak{m}$ -primary ideal, it contains a system of parameters, say,  $y_1, \dots, y_d$ . Then, filtering by the powers of  $y_i$ , we obtain that

$$\lambda(R/(I, x_1, \dots, x_h)^{[q]}) \leq \lambda(R/(y_1, \dots, y_d)^{[q]}) \leq q^d \lambda(R/(y_1, \dots, y_d)).$$

Last, let  $C = \lambda(R/(y_1, \dots, y_d))$  and observe that  $d = \dim R$  and  $h = \dim R/I$ .  $\square$

**Corollary 4.2.** *Let  $(R, \mathfrak{m})$  be a local ring, let  $J$  be an  $\mathfrak{m}$ -primary ideal, and  $I$  be an arbitrary ideal. If  $M$  is a finitely generated  $R$ -module, then there exists a constant  $D$  (independent of  $q'$ ) such that for all  $q, q'$*

$$\lambda(M/(I^{[q]} + J^{[qq']})M) \leq Dq'^{\dim R/I} q^{\dim M}.$$

*Proof.* Since  $J$  is  $\mathfrak{m}$ -primary,  $\mathfrak{m}^{[q_0]} \subseteq J$  for some  $q_0$ , thus if the result holds for  $J = \mathfrak{m}$

$$\lambda(M/(I^{[q]} + J^{[qq']})M) \leq \lambda(M/(I^{[q]} + \mathfrak{m}^{[qq'q_0]})M) \leq (Dq_0^{\dim M})q'^{\dim R/I} q^{\dim M}.$$

Therefore we assume that  $J = \mathfrak{m}$ .

Let  $K$  be the annihilator of  $M$  and let  $n$  be the minimal number of generators of  $M$ . Then there exists a surjection  $(R/K)^n \rightarrow M \rightarrow 0$ , so, after tensoring with  $R/(I^{[q]} + \mathfrak{m}^{[qq']})$ , we obtain from Lemma 4.1 the estimate

$$\begin{aligned} \lambda(M/(I^{[q]} + \mathfrak{m}^{[qq']})M) &\leq n \lambda(R/(K + I^{[q]} + \mathfrak{m}^{[qq']})) \leq nCq'^{\dim R/(I+K)} q^{\dim(R/K)} \\ &\leq nCq'^{\dim R/I} q^{\dim M}. \end{aligned}$$

$\square$

*Remark 4.3.* Suppose for two  $R$ -modules,  $M$  and  $N$ ,  $M_P \cong N_P$  for any prime  $P \in \text{Minh}(R)$ , i.e. any  $P$  such that  $\dim R/P = \dim R$ . If  $R$  is reduced, we want to observe that  $S^{-1}M \cong S^{-1}N$  where  $S$  is the complement to the union of all primes in  $\text{Minh}(R)$ . This follows from the isomorphism  $S^{-1}R \cong \prod_{P \in \text{Minh}(R)} R_P$ .

**Lemma 4.4.** *Let  $(R, \mathfrak{m})$  be a local ring, let  $J$  be an  $\mathfrak{m}$ -primary ideal, and  $I$  be an arbitrary ideal. Let  $M, N$  be finitely generated  $R$ -modules. Moreover, suppose  $M_P \cong N_P$  for any minimal prime  $P$  such that  $\dim R/P = \dim R$ . Then there exists a constant  $C$  independent of  $q'$  and such that for all  $q, q'$*

$$|\lambda(M/(I^{[q]} + J^{[qq']})M) - \lambda(N/(I^{[q]} + J^{[qq']})N)| < Cq'^{\dim R/I} q^{\dim R-1}.$$

*Proof.* Let  $S = R \setminus \bigcup_{P \in \text{Minh}(R)} P$ . By the previous remark,  $S^{-1}M = S^{-1}N$ , so, since  $S^{-1} \text{Hom}_R(M, N) = \text{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ , there exist homomorphisms  $M \rightarrow N$  and  $N \rightarrow M$  that become isomorphisms after localization by  $S$ . Thus we have exact sequences

$$M \rightarrow N \rightarrow K_1 \rightarrow 0,$$

$$N \rightarrow M \rightarrow K_2 \rightarrow 0$$

where  $\dim K_1, \dim K_2 < \dim R$ , since  $S^{-1}K_1 = S^{-1}K_2 = 0$ .

Tensoring the first exact sequence with  $R/(I^{[q]} + J^{[qq']})$  and taking length, we obtain that

$$\lambda(N/(I^{[q]} + J^{[qq']})N) \leq \lambda(M/(I^{[q]} + J^{[qq']})M) + \lambda(K_1/(I^{[q]} + J^{[qq']})K_1),$$

while the second sequence yields

$$\lambda(M/(I^{[q]} + J^{[qq']})M) \leq \lambda(N/(I^{[q]} + J^{[qq']})N) + \lambda(K_2/(I^{[q]} + J^{[qq']})K_2).$$

By Corollary 4.2, there are constants  $C_1$  and  $C_2$  such that  $\lambda(K_1/(I^{[q]} + J^{[qq']})K_1) \leq C_1 q'^{\dim R/I} q^{\dim R-1}$  and  $\lambda(K_2/(I^{[q]} + J^{[qq']})K_2) \leq C_2 q'^{\dim R/I} q^{\dim R-1}$ . Combining the estimates together, we derive that

$$|\lambda(N/(I^{[q]} + J^{[qq']})N) - \lambda(M/(I^{[q]} + J^{[qq']})M)| \leq \max(C_1, C_2) q'^{\dim R/I} q^{\dim R-1}.$$

□

For the next result, we need a bit of notation. For an  $R$ -module  $M$  and an integer  $e$ , we use  $F_*^e M$  to denote an  $R$ -module obtained from  $M$  via the extension of scalars through the  $e$ th iterate of the Frobenius endomorphism  $F^e: R \rightarrow R$ . Thus  $F_*^e M$  is isomorphic to  $M$  as an abelian group, but elements of  $R$  act as  $p^e$ -powers. So, for any ideal  $I$ ,  $IF_*^e M \cong F_*^e I^{[q]} M$ . If  $R$  is reduced,  $F_* R$  can be identified with the ring of  $p$ -roots  $R^{1/p}$ .

**Definition 4.5.** Let  $R$  be a ring of characteristic  $p > 0$ . For a prime ideal  $\mathfrak{p}$  of  $R$ , we denote  $\alpha(\mathfrak{p}) = \log_p[k(\mathfrak{p}) : k(\mathfrak{p})^p]$ , where  $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$  is the residue field of  $\mathfrak{p}$ .

This number controls the change of length under Frobenius,  $\lambda_R(F_*^e M) = p^{e\alpha(R)} \lambda_R(M)$  for a finite length  $R$ -module  $M$ .

Kunz ([15, Proposition 2.3]) observed how these numbers change under localization.

**Proposition 4.6.** Let  $R$  be  $F$ -finite and let  $\mathfrak{p} \subseteq \mathfrak{q}$  be prime ideals. Then  $\alpha(\mathfrak{p}) = \alpha(\mathfrak{q}) + \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ .

**Corollary 4.7.** Let  $(R, \mathfrak{m})$  be a reduced  $F$ -finite local ring of dimension  $d$  and  $M$  be a finitely generated  $R$ -module. Then for any minimal prime  $\mathfrak{p} \in \text{Minh}(R)$  of  $R$  the modules  $M_{\mathfrak{p}}^{\oplus p^{\alpha(R)+d}}$  and  $(F_* M)_{\mathfrak{p}}$  are isomorphic.

In particular, if  $R$  is equidimensional and  $M$  has rank  $r$ , then the rank of  $F_* M$  is  $rp^{\alpha(R)+d}$ .

*Proof.* By Proposition 4.6,  $\alpha(\mathfrak{p}) = \alpha(R) + d$  for any  $\mathfrak{p} \in \text{Minh}(R)$ . Hence

$$\lambda_{R_{\mathfrak{p}}}(F_*(M_{\mathfrak{p}})) = p^{\alpha(\mathfrak{p})} \lambda_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Since  $R$  is reduced,  $R_{\mathfrak{p}}$  is a field, hence the vector spaces  $F_*(M_{\mathfrak{p}})$  and  $\oplus^{\alpha(R)+d} M_{\mathfrak{p}}$  are isomorphic. Last, recall that  $F_*(M_{\mathfrak{p}}) \cong (F_* M)_{\mathfrak{p}}$  since  $F_*$  commutes with localization. □

**Theorem 4.8.** Let  $(R, \mathfrak{m})$  be a reduced  $F$ -finite local ring of dimension  $d$ ,  $J$  an  $\mathfrak{m}$ -primary ideal, and  $I$  be an ideal. Then for any finitely generated  $R$ -module  $M$  there exists a constant  $C$  such that

$$|\lambda(M/(I^{[q]} + J^{[qq']})M) - q^d e_{\text{HK}}(I + J^{[q]}, M)| < C q^{d-1} q'^{\dim R/I}$$

for all  $q, q'$ . In particular, the bisequence

$$\frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}}$$

converges uniformly with respect to  $q$ .

*Proof.* By Corollary 4.7,  $(F_* M)_P$  and  $M_P^{\oplus p^{\alpha(R)+d}}$  are isomorphic for any minimal prime  $P \in \text{Minh}(R)$ . Thus, we can apply Lemma 4.4 to  $M^{\oplus p^{\alpha(R)+d}}$  and  $F_* M$  and get that

$$|p^{\alpha(R)+d} \lambda(M/(I^{[q]} + J^{[qq']})M) - \lambda(F_* M/(I^{[q]} + J^{[qq']})F_* M)| < C q'^{\dim R/I} q^{d-1}$$

for any  $q, q'$  and a constant  $C$  depending only on  $M$  and  $I$ . We have

$$\lambda(F_* M/(I^{[q]} + J^{[qq']})F_* M) = p^{\alpha(R)} \lambda(M/(I^{[qp]} + J^{[qpq']})M).$$

Therefore, using that  $p^{-\alpha(R)}C \leq C$ ,

$$(4.1) \quad |p^d \lambda(M/(I^{[q]} + J^{[qq']})M) - \lambda(M/(I^{[qp]} + J^{[qpq']})M)| < Cq'^{\dim R/I} q^{d-1}.$$

Now, we prove by induction on  $q_1$  that for all  $q, q'$

$$(4.2) \quad |(q_1)^d \lambda(M/(I^{[q]} + J^{[qq']})M) - \lambda(M/(I^{[qq_1]} + J^{[qq_1q']})M)| < Cq'^{\dim R/I} (qq_1/p)^{d-1} \frac{q_1 - 1}{p - 1}.$$

The induction base of  $q_1 = p$  is (4.1). Now, assume that the claim holds for  $q_1$  and we want to prove it for  $q_1 p$ .

First, (4.1) applied to  $qq_1$  gives

$$(4.3) \quad |p^d \lambda(M/(I^{[qq_1]} + J^{[qq_1q']})M) - \lambda(M/(I^{[qq_1p]} + J^{[qq_1pq']})M)| < Cq'^{\dim R/I} (qq_1)^{d-1},$$

and, multiplying the induction hypothesis by  $p^d$ , we get

$$(4.4) \quad |(q_1 p)^d \lambda(M/(I^{[q]} + J^{[qq']})M) - p^d \lambda(M/(I^{[qq_1]} + J^{[qq_1q']})M)| < Cq'^{\dim R/I} (qq_1)^{d-1} \frac{q_1 p - p}{p - 1}.$$

Using that  $p^d \lambda(M/(I^{[qq_1]} + J^{[qq_1q']})M)$  appears in both (4.3) and (4.4), one obtains

$$\begin{aligned} & |(q_1 p)^d \lambda(M/(I^{[q]} + J^{[qq']})M) - \lambda(M/(I^{[qq_1p]} + J^{[qq_1pq']})M)| < \\ & < Cq'^{\dim R/I} (qq_1)^{d-1} \left( \frac{q_1 p - p}{p - 1} + 1 \right) = Cq'^{\dim R/I} (qq_1)^{d-1} \left( \frac{q_1 p - 1}{p - 1} \right), \end{aligned}$$

and the induction step follows.

Now, dividing (4.2) by  $q_1^d$ , we obtain

$$|\lambda(M/(I^{[q]} + J^{[qq']})M) - \frac{1}{q_1^d} \lambda(M/(I^{[qq_1]} + J^{[qq_1q']})M)| < Cq^{d-1} \cdot \frac{q_1 - 1}{p - 1} \cdot \frac{1}{q_1 p^{d-1}} \leq Cq^{d-1}.$$

Thus, if we let  $q_1 \rightarrow \infty$  and note that  $e_{\text{HK}}(I^{[q]} + J^{[qq']}, M) = q^d e_{\text{HK}}(I + J^{[q']}, M)$ , we get that

$$|\lambda(M/(I^{[q]} + J^{[qq']})M) - q^d e_{\text{HK}}(I + J^{[q']}, M)| < Cq^{d-1},$$

and the claim follows.  $\square$

Even if  $R$  is not F-finite we can often pass to an F-finite ring via the following recipe.

*Remark 4.9.* Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$ .

By Cohen's structure theorem ([19, page 265]),  $\widehat{R}$  has a coefficient field  $k$ . So, if  $x_1, \dots, x_n$  are the generators of  $\widehat{\mathfrak{m}R}$ , then there is a surjection  $k[[T_1, \dots, T_n]] \rightarrow \widehat{R}$  where  $T_i$  are indeterminates and the map acts by  $T_i \mapsto x_i$ .

Let  $S = \widehat{R} \otimes_{k[[T_1, \dots, T_n]]} k^\infty[[T_1, \dots, T_n]]$ , where  $k^\infty$  is the perfect closure of  $k$ . Then  $R \rightarrow S$  is faithfully flat as a composition of faithfully flat maps  $R \rightarrow \widehat{R} \rightarrow S$ . Moreover,  $S$  is complete with a perfect residue field, so, it is F-finite as a homomorphic image of the power series ring over a perfect field.

Also, note that  $\mathfrak{m}S$  is the maximal ideal of  $S$ . Thus  $\otimes_R S$  preserves length, and therefore  $e_{\text{HK}}(I, M) = e_{\text{HK}}(I, S \otimes_R M)$  for any  $\mathfrak{m}$ -primary ideal  $I$  and finitely generated module  $M$ .

**Corollary 4.10.** *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d$ ,  $J$  an  $\mathfrak{m}$ -primary ideal, and  $I$  be an arbitrary ideal. Then there is a  $q_0$  such that for any finitely generated  $R$ -module  $M$  there exists a constant  $C$  such that*

$$|\lambda(M/(I^{[q]} + J^{[qq']})M) - q^d e_{\text{HK}}(I + J^{[q']}, M)| < Cq^{d-1} q'^{\dim R/I}$$

for all  $q'$  and all  $q \geq q_0$ . In particular, the bisequence

$$\frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}}$$

converges uniformly with respect to  $q$ .

*Proof.* First, we reduce to the case where  $R$  is F-finite. Using the recipe in Remark 4.9, we can find a faithfully flat F-finite extension  $S$  of  $R$  such that  $\mathfrak{m}S$  is the maximal ideal of  $S$ . Hence, for any Artinian  $R$ -module  $A$ ,  $\lambda_S(A \otimes_R S) = \lambda_R(A)$ .

Now, there is  $q_0 = p^{e_0}$  such that  $(\sqrt{0}S)^{q_0} = 0$ . Naturally,  $N = F_*^{e_0}(S \otimes_R M)$  is a  $S_{red}$ -module, where  $S_{red} = S/\sqrt{0}S$ . Since  $S_{red}$  is reduced and F-finite, we can apply Theorem 4.8 and find a constant  $C$  such that

$$(4.5) \quad |\lambda_S(S_{red}/(I^{[q]} + J^{[qq']})S_{red} \otimes_{S_{red}} N) - q^d e_{HK}((I + J^{[q']})S_{red}, N)| < Cq^{d-1}$$

for all  $q, q'$ .

Now, observe that

$$\begin{aligned} \frac{S_{red}}{(I^{[q]} + J^{[qq']})S_{red}} \otimes_{S_{red}} N &\cong \frac{S}{(I^{[q]} + J^{[qq']})S} \otimes_S N \cong F_*^{e_0} \left( \frac{S \otimes_R M}{(I^{[qq_0]} + J^{[qq_0q']})(S \otimes_R M)} \right) \\ &\cong F_*^{e_0} \left( \frac{M}{(I^{[qq_0]} + J^{[qq_0q']})M} \otimes_R S \right). \end{aligned}$$

So,

$$\lambda_S \left( \frac{N}{(I^{[q]} + J^{[qq']})N} \right) = \lambda_S \left( \frac{M}{(I^{[qq_0]} + J^{[qq_0q']})M} \otimes_R S \right) = \lambda_R(M/(I^{[qq_0]} + J^{[qq_0q']})M).$$

Therefore, by definition,

$$e_{HK}((I + J^{[q']})S_{red}, N) = e_{HK}((I + J^{[q']})^{[q_0]}, M) = q_0^d e_{HK}(I + J^{[q']}, M)$$

and we can rewrite (4.5) as

$$|\lambda_R(M/(I^{[qq_0]} + J^{[qq_0q']})M) - (qq_0)^d e_{HK}(I + J^{[q']}, M)| < Cq^{d-1} \leq C(qq_0)^{d-1}.$$

By setting  $q = qq_0$ , we get that for all  $q \geq q_0$  and all  $q'$

$$|\lambda_R(M/(I^{[q]} + J^{[qq']})M) - q^d e_{HK}(I + J^{[q']}, M)| < Cq^{d-1}.$$

□

Now, we can establish the main result of this section, which will be the basic tool of our theory of equimultiplicity.

**Corollary 4.11.** *Let  $(R, \mathfrak{m})$  be a local ring, and  $J$  be an  $\mathfrak{m}$ -primary ideal. If  $I$  is an ideal such that  $\dim R/I + \text{ht } I = \dim R$ , then*

$$\lim_{q' \rightarrow \infty} e_{HK}(I + J^{[q']}, M) = \sum_{P \in \text{Minh}(I)} e_{HK}(JR/P, R/P) e_{HK}(IR_P, M_P).$$

*Proof.* We have proved that the double sequence

$$\frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}}$$

converges uniformly with respect to  $q$ . Moreover, the limit with respect to  $q'$  exists for any  $q$  since

$$\lim_{q' \rightarrow \infty} \frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}} = \frac{e_{HK}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)}{q^{\dim R}},$$

where  $e_{\text{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)$  is taken over the ring  $R/I^{[q]}$ . Thus, the iterated limits of the double sequence are equal:

$$\begin{aligned} \lim_{q' \rightarrow \infty} \frac{e_{\text{HK}}(I + J^{[q']}, M)}{q'^{\dim R/I}} &= \lim_{q' \rightarrow \infty} \lim_{q \rightarrow \infty} \frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}} \\ &= \lim_{q \rightarrow \infty} \lim_{q' \rightarrow \infty} \frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}} = \lim_{q \rightarrow \infty} \frac{e_{\text{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)}{q^{\dim R}}. \end{aligned}$$

Note that  $\sqrt{I} = \sqrt{I^{[q]}}$ , so  $\dim R/I = \dim R/I^{[q]}$  and  $\text{Minh}(I) = \text{Minh}(I^{[q]})$ . Moreover, by the associativity formula,

$$e_{\text{HK}}(JR/I^{[q]}, M/I^{[q]}M) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(JR/P, R/P) \lambda_{R_P}(M_P/I^{[q]}M_P).$$

Hence, we may use that  $e_{\text{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M) = q^{\dim R/I} e_{\text{HK}}(JR/I^{[q]}, M/I^{[q]}M)$  and obtain that

$$\begin{aligned} \lim_{q \rightarrow \infty} \frac{e_{\text{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)}{q^{\dim R}} &= \lim_{q \rightarrow \infty} \frac{e_{\text{HK}}(JR/I^{[q]}, M/I^{[q]}M)}{q^{\text{ht } I}} \\ &= \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(JR/P, R/P) \lim_{q \rightarrow \infty} \frac{\lambda_{R_P}(M_P/I^{[q]}M_P)}{q^{\text{ht } I}}. \end{aligned}$$

The claim follows, since  $\text{ht } I = \text{ht } P$ . □

**Corollary 4.12.** *Let  $(R, \mathfrak{m})$  be a local ring,  $J$  an  $\mathfrak{m}$ -primary ideal, and  $\mathfrak{p}$  be a prime ideal such that  $\dim R/\mathfrak{p} + \text{ht } \mathfrak{p} = \dim R$ . Then*

$$\lim_{q \rightarrow \infty} \frac{e_{\text{HK}}(\mathfrak{p} + J^{[q]})}{q^{\dim R/\mathfrak{p}}} = e_{\text{HK}}(JR/\mathfrak{p}, R/\mathfrak{p}) e_{\text{HK}}(R_{\mathfrak{p}}).$$

When  $R/\mathfrak{p}$  is regular, this corollary will help us to connect  $e_{\text{HK}}(\mathfrak{m})$  to  $e_{\text{HK}}(\mathfrak{p})$ .

## 5. EQUIMULTIPLICITY THEORY

In this section, we study equimultiple ideals for Hilbert-Kunz multiplicity. We will find that these should be ideals  $I$  such that for any (or, as we will show, some) system of parameters  $J = (x_1, \dots, x_m)$  modulo  $I$

$$(5.1) \quad e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P).$$

This could be seen as a direct analogue of condition (b) of Theorem 1.1.

Before proceeding further let us record the following lemma that shows that Equation 5.1 is an extremal condition. Equimultiplicity is also an extremal condition as it is given by the following result of Kunz ([15, Proposition 3.3]).

**Proposition 5.1.** *Let  $(R, \mathfrak{m})$  be an equidimensional catenary ring. Then any prime ideal  $\mathfrak{p}$*

$$e_{\text{HK}}(\mathfrak{p}) \leq e_{\text{HK}}(\mathfrak{m}).$$

**Lemma 5.2.** *Let  $(R, \mathfrak{m})$  to be a local ring of characteristic  $p > 0$  and let  $I$  be an ideal such that  $\text{ht } I + \dim R/I = \dim R$ . Then for any parameter ideal  $J$  modulo  $I$*

$$e_{\text{HK}}(I + J) \geq \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(IR_P).$$

*Proof.* First, by Proposition 2.15 and Corollary 2.17,

$$\lambda(R/(I+J)^{[q]}) \geq e(J^{[q]}, R/I^{[q]}) = q^{\dim R/I} e(J, R/I^{[q]}).$$

So, by the associativity formula,

$$e_{\text{HK}}(I+J) \geq \lim_{q \rightarrow \infty} \sum_{P \in \text{Minh}(I)} \frac{1}{q^{\text{ht } I}} e(J, R/P) \lambda(R_P/I^{[q]} R_P)$$

and the claim follows.  $\square$

**5.1. Preliminaries.** Let  $x_1, \dots, x_m \in R$  be a system of parameters modulo  $I$ , i.e., the images of  $x_1, \dots, x_m$  form system of parameters in  $R/I$ . To make our notation less cumbersome, in the following we are going to write  $e_{\text{HK}}(I, x_1, \dots, x_m)$  instead of  $e_{\text{HK}}((I, x_1, \dots, x_m))$ . The main ingredient of our proof is the multi-sequence

$$\lim_{\min(n_i) \rightarrow \infty} \frac{1}{n_1 \cdots n_m} e_{\text{HK}}(I, x_1^{n_1}, \dots, x_m^{n_m})$$

that will allow us to connect the two sides of Equation 5.1.

First, let us record two useful result in the case when  $\dim R/I = 1$ .

**Lemma 5.3.** *Let  $(R, \mathfrak{m})$  be a local ring,  $I$  be an ideal, and  $x$  be a parameter modulo  $I$ . Suppose  $\dim R/I = \dim R - 1$ . Then*

$$\frac{1}{n} \lambda(R/(I, x^n)) \geq \frac{1}{n+1} \lambda(R/(I, x^{n+1})).$$

*Proof.* Observe that

$$\frac{I + (x^k)}{I + (x^{k+1})} \cong \frac{(x^k)}{(x^k) \cap I + (x^{k+1})} \cong \frac{R}{I : x^k + (x)}.$$

Thus we get the formula

$$\lambda\left(\frac{R}{(I, x^{k+1})}\right) = \lambda\left(\frac{R}{(I, x^k)}\right) + \lambda\left(\frac{R}{I : x^k + (x)}\right).$$

First of all, setting  $n = k$  in the formula, we see that it is enough to show that  $\lambda(R/(I, x^n)) \geq n \lambda(R/(I : x^n + (x)))$ . Second, using the formula above for consecutive values of  $k$ , we obtain that

$$\lambda(R/(I, x^n)) = \lambda(R/(I, x)) + \sum_{k=1}^{n-1} \lambda(R/(I : x^k + (x))) \geq n \lambda(R/(I : x^n + (x))),$$

where the last inequality holds since  $I : x^k \subseteq I : x^n$  for all  $0 \leq k \leq n$ .  $\square$

**Corollary 5.4.** *In the setting of the lemma, we have*

$$\frac{1}{n} e_{\text{HK}}(I, x^n) \geq \frac{1}{n+1} e_{\text{HK}}(I, x^{n+1}).$$

*Proof.* Apply the lemma to  $I^{[q]}$  and  $x^q$  and take the limit as  $q \rightarrow \infty$ .  $\square$

Now, we can show that our multi-sequence converges to the right-hand side of Equation 5.1.

**Proposition 5.5.** *Let  $(R, \mathfrak{m})$  to be a local ring of characteristic  $p > 0$  and let  $I$  be an ideal in  $R$  such that  $\text{ht } I = \dim R - \dim R/I$ . If  $x_1, \dots, x_m$  are a system of parameters modulo  $I$  then*

$$\lim_{\min(n_i) \rightarrow \infty} \frac{1}{n_1 \cdots n_m} e_{\text{HK}}(I, x_1^{n_1}, \dots, x_m^{n_m}) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}((x_1, \dots, x_m), R/P) e_{\text{HK}}(IR_P).$$

*Proof.* Let  $(n_1, \dots, n_m) \in \mathbb{N}^m$  be an arbitrary vector and let  $n = \min(n_1, \dots, n_m)$  and  $N = \max(n_1, \dots, n_m)$ . Corollary 5.4 shows that

$$\frac{1}{N^m} e_{\text{HK}}(I, x_1^N, \dots, x_m^N) \leq \frac{1}{n_1 \cdots n_m} e_{\text{HK}}(I, x_1^{n_1}, \dots, x_m^{n_m}) \leq \frac{1}{n^m} e_{\text{HK}}(I, x_1^n, \dots, x_m^n).$$

Therefore,

$$\lim_{\min(n_i) \rightarrow \infty} \frac{1}{n_1 \cdots n_m} e_{\text{HK}}(I, x_1^{n_1}, \dots, x_m^{n_m}) = \lim_{n \rightarrow \infty} \frac{1}{n^m} e_{\text{HK}}(I, x_1^n, \dots, x_m^n).$$

Moreover, by Corollary 5.4, the sequence  $\frac{1}{n^m} e_{\text{HK}}(I, x_1^n, \dots, x_m^n)$  is monotonically decreasing, so its limit exists and can be computed by looking at a subsequence. But, by Corollary 4.11,

$$\lim_{q' \rightarrow \infty} \frac{1}{q'^m} e_{\text{HK}}(I, x_1^{q'}, \dots, x_m^{q'}) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}((x_1, \dots, x_m), R/P) e_{\text{HK}}(IR_P).$$

□

**Corollary 5.6.** *In the assumptions of Proposition 5.5, if*

$$e_{\text{HK}}(I, x_1, \dots, x_m) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}((x_1, \dots, x_m), R/P) e_{\text{HK}}(IR_P),$$

*then for any vector  $(n_1, \dots, n_m) \in \mathbb{N}^m$*

$$e_{\text{HK}}(I, x_1^{n_1}, \dots, x_m^{n_m}) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}((x_1^{n_1}, \dots, x_m^{n_m}), R/P) e_{\text{HK}}(IR_P).$$

*Proof.* By Corollary 5.4,

$$e_{\text{HK}}(I, x_1, \dots, x_m) \geq \frac{e_{\text{HK}}(I, x_1^{n_1}, \dots, x_m^{n_m})}{n_1 \cdots n_m}.$$

Moreover, by Lemma 5.2 and Corollary 2.17,

$$\begin{aligned} \frac{e_{\text{HK}}(I, x_1^{n_1}, \dots, x_m^{n_m})}{n_1 \cdots n_m} &\geq \sum_{P \in \text{Minh}(I)} \frac{e_{\text{HK}}((x_1^{n_1}, \dots, x_m^{n_m}), R/P)}{n_1 \cdots n_m} e_{\text{HK}}(IR_P) \\ &= \sum_{P \in \text{Minh}(I)} e_{\text{HK}}((x_1, \dots, x_m), R/P) e_{\text{HK}}(IR_P) = e_{\text{HK}}(I, x_1, \dots, x_m). \end{aligned}$$

□

For the next corollary recall the following observation ([30, Lemma 4.2]).

**Lemma 5.7.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  and  $I$  an  $\mathfrak{m}$ -primary ideal. Then  $e_{\text{HK}}(I) \leq \lambda(R/I) e_{\text{HK}}(\mathfrak{m})$ .*

**Corollary 5.8.** *Let  $(R, \mathfrak{m})$  to be a local ring of characteristic  $p > 0$  and  $\mathfrak{p}$  be a prime ideal in  $R$  such that  $R/\mathfrak{p}$  is a regular local ring and  $\text{ht } \mathfrak{p} = \dim R - \dim R/\mathfrak{p}$ . Then the following are equivalent:*

- (i)  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$ ,
- (ii)  $e_{\text{HK}}(\mathfrak{p}, x_1^{n_1}, \dots, x_m^{n_m}) = n_1 \cdots n_m e_{\text{HK}}(\mathfrak{m})$  for all vectors  $(n_1, \dots, n_m)$  and any system of minimal generators  $x_1, \dots, x_n$  of  $\mathfrak{m}$  modulo  $\mathfrak{p}$ ,
- (iii)  $e_{\text{HK}}(\mathfrak{p} + I) = \lambda(R/(\mathfrak{p} + I)) e_{\text{HK}}(\mathfrak{m})$  for any system of parameters  $I$  modulo  $\mathfrak{p}$ .

*Proof.* Clearly, (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i). For (i)  $\Rightarrow$  (iii), we observe by Lemma 5.2 that  $e_{\text{HK}}(\mathfrak{p} + I) \geq \lambda(R/(\mathfrak{p} + I)) e_{\text{HK}}(\mathfrak{p})$ . On the other hand, the preceding remark gives that

$$e_{\text{HK}}(\mathfrak{p} + I) \leq \lambda(R/(\mathfrak{p} + I)) e_{\text{HK}}(\mathfrak{m}) = \lambda(R/(\mathfrak{p} + I)) e_{\text{HK}}(\mathfrak{p}).$$

□



After building some machinery, we are going to show (Corollary 5.18) that the equality  $e_{\text{HK}}(\mathfrak{p} + I) = \lambda(R/(\mathfrak{p} + I)) e_{\text{HK}}(\mathfrak{p})$  for some system of parameters  $I$  forces that  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$ .

**5.2. Main results.** The next fundamental theorem can be seen as an analogue of implication  $(b) \Rightarrow (d)$  of Theorem 1.1.

**Theorem 5.9.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring of characteristic  $p > 0$  with a test element  $c$ . Let  $I$  be an ideal and suppose for some system of parameters  $J = (x_1, \dots, x_m)$  modulo  $I$ ,*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P).$$

*Then  $(I, x_1, \dots, x_{i-1})^{[q]} : x_i^\infty \subseteq \left( (I, x_1, \dots, x_{i-1})^{[q]} \right)^*$  for all  $q$  and  $1 \leq i \leq m$ .*

*In particular,  $x_i$  is not a zero divisor modulo  $\left( (I, x_1, \dots, x_{i-1})^{[q]} \right)^*$  for all  $i$  and  $q$ .*

*Proof.* First, observe that the second claim follow from the first via Lemma 2.12.

Let  $d$  be the dimension of  $R$ . For a fixed  $k$ , let  $L = (x_1, x_2, \dots, x_{i-1}, x_{i+1}^k, \dots, x_m^k)$ . For any  $n, q, q'$  we have inclusions

$$(5.2) \quad (I, L, x_i^k)^{[qq']} \subseteq \left( (I, x_1, \dots, x_{i-1})^{[q]} : x_i^{nq} \right)^{[q']} + (L, x_i^k)^{[qq']} \subseteq (I, L)^{[qq']} : x_i^{nqq'} + (x_i^{kqq'})^{[q]}.$$

Hence, after dividing by  $q'^d$  and taking the limit, we obtain that

$$(5.3) \quad \begin{aligned} e_{\text{HK}} \left( (I, L, x_i^k)^{[q]} \right) &\geq e_{\text{HK}} \left( (I, x_1, \dots, x_{i-1})^{[q]} : x_i^{nq} + (L, x_i^k)^{[q]} \right) \\ &\geq \lim_{q' \rightarrow \infty} \frac{1}{(q')^d} \lambda \left( \frac{R}{(I, L)^{[qq']} : x_i^{nqq'} + (x_i^{kqq'})^{[q]}} \right). \end{aligned}$$

By Corollary 5.6 and Corollary 2.17, for all  $n$

$$e_{\text{HK}}(I, L, x_i^n) = n \sum_{P \in \text{Minh}(I)} e((x_i, L), R/P) e_{\text{HK}}(I, R_P).$$

In particular, we obtain that

$$e_{\text{HK}} \left( (I, L, x_i^k)^{[q]} \right) = q^d e_{\text{HK}}(I, L, x_i^k) = kq^d \sum_{P \in \text{Minh}(I)} e((x_i, L), R/P) e_{\text{HK}}(I, R_P).$$

Moreover, from the isomorphism  $R / \left( (I, L)^{[q]} : x_i^{nq}, x_i^{kq} \right) \cong (I, L, x_i^n)^{[q]} / (I, L, x_i^{n+k})^{[q]}$ , we get an exact sequence

$$0 \rightarrow R / \left( (I, L)^{[q]} : x_i^{nq}, x_i^{kq} \right) \rightarrow R / (I, L, x_i^{n+k})^{[q]} \rightarrow R / (I, L, x_i^n)^{[q]} \rightarrow 0.$$

Together with the previous computation, the sequence gives that for all  $n$  and  $k$

$$\lim_{q \rightarrow \infty} \frac{1}{q^d} \lambda \left( \frac{R}{(I, L)^{[q]} : x_i^{nq} + (x_i^{kq})^{[q]}} \right) = k \sum_{P \in \text{Minh}(I)} e((x_i, L), R/P) e_{\text{HK}}(I, R_P),$$

so, we may compute

$$\lim_{(qq') \rightarrow \infty} \frac{q^d}{(qq')^d} \lambda \left( \frac{R}{(I, L)^{[qq']} : x_i^{nqq'} + (x_i^{kqq'})^{[q]}} \right) = kq^d \sum_{P \in \text{Minh}(I)} e((x_i, L), R/P) e_{\text{HK}}(I, R_P).$$

Thus, by (5.2) and (5.3)

$$e_{\text{HK}}\left((I, x_1, \dots, x_{i-1})^{[q]} : x_i^{nq}, (L, x_i^k)^{[q]}\right) = e_{\text{HK}}\left((I, L, x_i^k)^{[q]}\right).$$

Therefore, by Theorem 2.7,  $(I, x_1, \dots, x_{i-1})^{[q]} : x_i^{nq} \subseteq \left((I, L, x_i^k)^{[q]}\right)^*$ . Now, since  $n$  is arbitrary, we have

$$(I, x_1, \dots, x_{i-1})^{[q]} : x_i^\infty \subseteq \bigcap_k \left((I, x_1, \dots, x_{i-1}, x_i^k, \dots, x_m^k)^{[q]}\right)^*$$

and the assertion follows from Lemmas 2.5 and 2.12.  $\square$

Now, we will establish the converse to Theorem 5.9. But first we will need the following definition.

**Definition 5.10.** Let  $R$  be a ring and  $c \in R^\circ$ . We say that  $c$  is a *locally stable test element* if the image of  $c$  in  $R_P$  is a test element for any prime  $P$ .

While this condition is stronger than that of a test element, in fact, the known results on existence of tests elements provide us locally stable test elements. In particular, Theorem 2.4 asserts that locally stable test elements exists for F-finite domains and algebras of essentially finite type over an excellent local domain.

**Theorem 5.11.** Let  $(R, \mathfrak{m})$  be a formally unmixed local ring of characteristic  $p > 0$  with a locally stable test element  $c$ . Let  $I$  be an ideal and  $J = (x_1, \dots, x_m)$  be a system of parameters modulo  $I$ . Then

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P)$$

if and only if  $x_i$  is not a zero divisor modulo  $((I, x_1, \dots, x_{i-1})^{[q]})^*$  for all  $i$  and  $q$ .

*Proof.* One direction follows from Theorem 5.9.

For the converse, we use induction on  $m$ . If  $m = 1$ , then by Proposition 2.15 and by the associativity formula (Proposition 2.14)

$$\lambda(R/((I^{[q]})^*, x_1^q)) = q e(x_1, R/(I^{[q]})^*) = q \sum_{P \in \text{Minh}(I)} e(x_1, R/P) \lambda(R_P/(I^{[q]})^* R_P).$$

So, taking the limit via Lemma 2.9,  $e_{\text{HK}}(I, x_1) \leq \sum_{P \in \text{Minh}(I)} e(x_1, R/P) e_{\text{HK}}(I, R_P)$  and the converse holds by Lemma 5.2.

Now, by the induction hypothesis,

$$e_{\text{HK}}(I + J) = e_{\text{HK}}((I, x_1) + (x_2, \dots, x_m)) = \sum_{Q \in \text{Minh}((I, x_1))} e_{\text{HK}}((x_2, \dots, x_m), R/Q) e_{\text{HK}}((I, x_1), R_Q).$$

Since  $c$  is locally stable, by Corollary 2.10  $((I^{[q]})^*, x_1^q) R_Q$  still can be used to compute  $e_{\text{HK}}((I, x_1), R_Q)$ . Thus, same way as in the first step, we obtain

$$e_{\text{HK}}((I, x_1), R_Q) = \sum_{P \in \text{Minh}(IR_Q)} e(x_1, R_Q/PR_Q) e_{\text{HK}}(I, R_P).$$

Combining these results, we get

$$e_{\text{HK}}(I + J) = \sum_{Q \in \text{Minh}((I, x_1))} e((x_2, \dots, x_m), R/Q) \sum_{P \in \text{Minh}(IR_Q)} e(x_1, R_Q/PR_Q) e_{\text{HK}}(I, R_P).$$

Observe that  $\text{Minh}(IR_Q) = \text{Min}(IR_Q)$ , since  $x_1$  is a parameter modulo  $I$  and  $R_Q/IR_Q$  has dimension 1. Hence, any prime  $P \in \text{Minh}(IR_Q)$  is just a minimal prime of  $I$  contained in  $Q$ .

Therefore, we can change the order of summations to get

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Min}(I)} e_{\text{HK}}(I, R_P) \sum_{P \subset Q \in \text{Minh}((I, x_1))} e(x_1, R_Q/PR_Q) e((x_2, \dots, x_m), R/Q),$$

where the second sum is taken over all primes  $Q \in \text{Minh}(I + (x_1))$  that contain  $P$ . For such  $Q$  we must have  $\dim R/P \geq \dim R/Q + 1 = \dim R/I$ , because  $x_1$  is a parameter modulo  $I$ . So, in fact, the first sum could be taken over  $\text{Minh}(I)$ . Furthermore, since  $x_1$  is a parameter modulo  $I$  and  $P \in \text{Minh}(I)$ ,  $x_1$  is a parameter modulo  $P$  and  $\dim R/(I, x_1) = \dim R/(P, x_1)$ . Hence, the second sum is taken over  $Q \in \text{Minh}((P, x_1))$ , and we rewrite the formula as

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(I, R_P) \sum_{Q \in \text{Minh}((P, x_1))} e(x_1, R_Q/PR_Q) e((x_2, \dots, x_m), R/Q).$$

Last, by the associativity formula for the multiplicity of a parameter ideal (Proposition 2.18), for any  $P$

$$e_{\text{HK}}(J, R/P) = \sum_{Q \in \text{Minh}((P, x_1))} e((x_2, \dots, x_m), R/Q) e(x_1, R_Q/PR_Q),$$

and the claim follows.  $\square$

**Corollary 5.12.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring of positive characteristic with a locally stable test element  $c$ . Let  $I$  be an ideal and  $J = (x_1, \dots, x_m)$  be a system of parameters modulo  $I$ . If*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P)$$

then for any  $0 \leq k \leq m$

$$e_{\text{HK}}(I + J) = \sum_{Q \in \text{Minh}((I, x_1, \dots, x_k))} e_{\text{HK}}((x_{k+1}, \dots, x_m), R/P) e_{\text{HK}}((I, x_1, \dots, x_k), R_Q).$$

*Proof.* First, by Theorem 5.9,  $x_i$  is not a zero divisor modulo  $((I, x_1, \dots, x_{i-1})^{[q]})^*$  for all  $i$  and  $q$ . Now, since this holds for all  $i \geq k$ , Theorem 5.11 shows the assertion.  $\square$

**Corollary 5.13.** *Let  $(R, \mathfrak{m})$  be a local ring and let  $x_1, \dots, x_d$  and  $y_1, \dots, y_d$  be two systems of parameters. Then there exists a linear combination  $x' = x_d + a_1 x_1 + \dots + a_{d-1} x_{d-1}$  with coefficients in  $R$  such that  $x_1, \dots, x_{d-1}, x'$  and  $y_1, \dots, y_{d-1}, x'$  are systems of parameters.*

*Proof.* First, it is easy to see that  $x_1, \dots, x_{d-1}, x'$  is still a system of parameters for any choice of the coefficients  $a_i$ .

Second, we let  $P_1, \dots, P_n$  be the minimal primes of  $(y_1, \dots, y_{d-1})$  and apply the avoidance lemma for cosets to  $x = x_d$  and  $I = (x_1, \dots, x_{d-1})$ .  $\square$

After all the hard work, we can establish that our definition of an equimultiple ideal is independent on the choice of a parameter ideal.

**Proposition 5.14.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring of characteristic  $p > 0$  with a locally stable test element  $c$  and let  $I$  be an ideal. If for some system of parameters  $J = (x_1, \dots, x_m)$  modulo  $I$*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P),$$

then same is true for all systems of parameters.

*Proof.* We use induction on  $m$  and Theorem 5.11. If  $\dim R/I = 1$ , then, by Theorem 5.9, our assumption shows that  $R/(I^{[q]})^*$  is Cohen-Macaulay for any  $q$ , so any parameter is regular.

Let  $(y_1, \dots, y_m)$  be an arbitrary system of parameters modulo  $I$ . Then using Corollary 5.13, we can find an element of the form  $x' = x_d + a_1x_1 + \dots + a_{m-1}x_{m-1}$  such that  $y_1, \dots, y_{m-1}, x'$  is still a system of parameters modulo  $I$ . Note that  $(x_1, \dots, x_{m-1}, x') = (x_1, \dots, x_m) = J$ , so the original formula still holds. By Corollary 5.12, we get that

$$e_{\text{HK}}(I + J) = e_{\text{HK}}(I, x', x_1, \dots, x_{m-1}) = \sum_{Q \in \text{Minh}((I, x'))} e_{\text{HK}}((x_1, \dots, x_{m-1}), R/Q) e_{\text{HK}}((I, x'), R_Q),$$

so, by the induction hypothesis applied to  $(I, x')$ ,

$$e_{\text{HK}}(I, x', y_1, \dots, y_{m-1}) = \sum_{Q \in \text{Minh}((I, x'))} e_{\text{HK}}((y_1, \dots, y_{m-1}), R/Q) e_{\text{HK}}((I, x'), R_Q).$$

Using Theorem 5.9 on  $(I, x')$ , we get that  $y_i$  is regular modulo  $((I, x', y_1, \dots, y_{i-1})^{[q]})^*$  for any  $i$  and  $q$ . But since  $x'$  is also regular modulo  $(I^{[q]})^*$  for all  $q$ , Theorem 5.11 implies that

$$e_{\text{HK}}(I, x', y_1, \dots, y_{m-1}) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}((x', y_1, \dots, y_{m-1}), R/P) e_{\text{HK}}(I, R_P).$$

After permuting the sequence and using Theorem 5.9, we see that  $x'$  is not a zero divisor modulo  $((I, y_2, \dots, y_n)^{[q]})^*$  for all  $q$ . Now, again, both  $x'$  and  $y_m$  are parameters modulo  $((I, y_1, \dots, y_{m-1})^{[q]})^*$ , so  $y_m$  is regular too.  $\square$

Motivated by Proposition 5.14 and Theorem 5.11, we introduce the following definition.

**Definition 5.15.** Let  $(R, \mathfrak{m})$  be a local ring and let  $I$  be an ideal. We say that  $I$  satisfies *colon capturing*, if for every system of parameters  $x_1, \dots, x_m$  in  $R/I$ , for every  $0 \leq i < m$ , and every  $q$

$$((I, x_1, \dots, x_i)^{[q]})^* : x_{i+1} \subseteq ((I, x_1, \dots, x_i)^{[q]})^*.$$

A well-known result of tight closure theory asserts that under mild conditions, (0) satisfies colon capturing. We note that the tight closure is taken in  $R$ , so this property is different from colon capturing in  $R/I$ .

*Remark 5.16.* The colon capturing property asserts that any system of parameters in  $R/I$  is “regular up to tight closure” modulo  $I$ . So it is not very surprising that the condition can be checked for a single system of parameters.

With this definition, we can summarize our findings in an analogue of equivalence (b) and (d) of Theorem 1.1.

**Theorem 5.17.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring of characteristic  $p > 0$  with a locally stable test element  $c$  and let  $I$  be an ideal. Then the following are equivalent:*

- (1)  *$I$  satisfies colon capturing,*
- (2) *for some (equivalently, every) ideal  $J$  which is a system of parameters modulo  $I$ ,*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P).$$

*Proof.* This was proved in Theorem 5.11 and Proposition 5.14.  $\square$

In the special case of prime ideals with regular quotients we obtain the following characterization.

**Corollary 5.18.** *Let  $(R, \mathfrak{m})$  be an formally unmixed local ring of characteristic  $p > 0$  with a locally stable test element  $c$  and let  $\mathfrak{p}$  be a prime ideal such that  $R/\mathfrak{p}$  is a regular local ring. Then the following are equivalent:*

- (a)  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$ ,
- (b) For any (equivalently, some) system of parameters  $J$  modulo  $\mathfrak{p}$ 

$$e_{\text{HK}}(\mathfrak{p}, J) = \lambda(R/(\mathfrak{p}, J)) e_{\text{HK}}(\mathfrak{m}),$$
- (c) For any (equivalently, some) system of parameters  $J$  modulo  $\mathfrak{p}$ 

$$e_{\text{HK}}(\mathfrak{p}, J) = \lambda(R/(\mathfrak{p}, J)) e_{\text{HK}}(\mathfrak{p}),$$
- (d)  $\mathfrak{p}$  satisfies colon capturing.

*Proof.* The first two conditions are equivalent by Corollary 5.8, (a), (c), (e) are equivalent by the previous theorem.  $\square$

This theorem has a notable corollary. First, recall that a ring  $R$  of characteristic  $p > 0$  is called weakly F-regular if  $I^* = I$  for every ideal  $I$  in  $R$ . For example, any regular ring is weakly F-regular and direct summands of weakly F-regular rings are weakly F-regular.

**Corollary 5.19.** *Let  $(R, \mathfrak{m})$  be a weakly F-regular excellent local domain and let  $\mathfrak{p}$  be a prime ideal such that  $R/\mathfrak{p}$  is regular. Then  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$  if and only if the Hilbert-Kunz functions of  $R$  and  $R_{\mathfrak{p}}$  coincide.*

*Proof.* Since all ideals in  $R$  are tightly closed, from the preceding theorem we obtain that  $R/\mathfrak{p}^{[q]}$  is Cohen-Macaulay for all  $q$ . Hence the assertion follows from Proposition 3.1.  $\square$

*Remark 5.20.* In [18, Lemma 4.2, Remark 4.7] Linquan Ma observed that perfect ideals are equimultiple and, thus, satisfy the colon capturing property. This observation is a crucial ingredient of his recent work on Lech's conjecture ([18]).

**5.3. Further improvements.** We will develop some general reductions for the equimultiplicity condition and use them to generalize the obtained results.

First, we show that equimultiplicity can be checked modulo minimal primes.

**Lemma 5.21.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  and  $\mathfrak{p}$  be a prime ideal such that  $\text{ht } \mathfrak{p} + \dim R/\mathfrak{p} = \dim R$ . Then  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$  if and only if  $\text{Minh}(R) = \text{Minh}(R_{\mathfrak{p}})$  and  $e_{\text{HK}}(\mathfrak{m}R/P) = e_{\text{HK}}(\mathfrak{p}R/P)$  for any  $P \in \text{Minh}(R)$ .*

*In particular, if  $R$  is catenary, then  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$  if and only if  $P \subseteq \mathfrak{p}$  and  $e_{\text{HK}}(\mathfrak{m}R/P) = e_{\text{HK}}(\mathfrak{p}R/P)$  for all  $P \in \text{Minh}(R)$ .*

*Proof.* If  $Q \in \text{Minh}(R_{\mathfrak{p}})$ , by definition,  $\dim R_{\mathfrak{p}}/QR_{\mathfrak{p}} = \text{ht } \mathfrak{p}$ , so  $Q \in \text{Minh}(R)$  by the assumption on  $\mathfrak{p}$ . Moreover, if  $R$  is catenary, it is easy to check that, in fact,  $\text{Minh}(R_{\mathfrak{p}}) = \{P \in \text{Minh}(R) \mid P \subseteq \mathfrak{p}\}$ .

By the associativity formula we have:

$$e_{\text{HK}}(\mathfrak{m}) = \sum_{P \in \text{Minh}(R)} e_{\text{HK}}(\mathfrak{m}, R/P) \lambda(R_P),$$

and, also by Corollary 5.1,

$$e_{\text{HK}}(\mathfrak{p}) = \sum_{Q \in \text{Minh}(R_{\mathfrak{p}})} e_{\text{HK}}(\mathfrak{p}, R_{\mathfrak{p}}/QR_{\mathfrak{p}}) \lambda(R_Q) \leq \sum_{Q \in \text{Minh}(R_{\mathfrak{p}})} e_{\text{HK}}(\mathfrak{m}, R/Q) \lambda(R_Q).$$

Since the second sum is contained in the sum appearing in the expression for  $e_{\text{HK}}(\mathfrak{m})$ , the claim follows.  $\square$

The lemma can be easily generalized, but we decided to leave the special case for clarity. A more general lemma can be found right after the following easy corollary.

**Corollary 5.22.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  and  $\mathfrak{p}$  be a prime ideal such that  $\text{ht } \mathfrak{p} + \dim R/\mathfrak{p} = \dim R$ . Then  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$  if and only if  $e_{\text{HK}}(\mathfrak{m}R_{\text{red}}) = e_{\text{HK}}(\mathfrak{p}R_{\text{red}})$ .*

*Proof.* Since  $\text{Minh}(R) = \text{Minh}(R_{\text{red}})$ , this immediately follows from the previous lemma applied to  $R$  and  $R_{\text{red}}$ .  $\square$

**Lemma 5.23.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  and  $I$  be an ideal such that  $\text{ht } I + \dim R/I = \dim R$ . Let  $J$  be a system of parameters modulo  $I$ . Then*

$$e_{\text{HK}}(I + J) = \sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q)$$

*if and only if the following two conditions hold:*

- (a)  $\text{Minh}(I + P) \subseteq \text{Minh}(I)$  for all  $P \in \text{Minh}(R)$ ,
- (b)  $e_{\text{HK}}(I + J, R/P) = \sum_{Q \in \text{Minh}(IR/P)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q/PR_Q)$  for all  $P \in \text{Minh}(R)$ .

*Proof.* First, observe that if  $P \in \text{Minh}(R)$  and  $Q \in \text{Minh}(I)$  such that  $P \subseteq Q$ , then  $\dim R/I \geq \dim R/(I + P) \geq \dim R/Q = \dim R/I$ , so  $Q \in \text{Minh}(I + P)$  and the image of  $Q$  in  $R/P$  is in  $\text{Minh}(IR/P)$ . Moreover, in this case,  $\dim R/(I + P) = \dim R/I$ , so  $\text{Minh}(I + P) \subseteq \text{Minh}(I)$ . And the converse is also true: if  $\text{Minh}(I + P) \subseteq \text{Minh}(I)$  then  $P$  is contained in some  $Q \in \text{Minh}(I)$ .

By the associativity formula for  $e_{\text{HK}}(IR_Q)$

$$\sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q) = \sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) \sum_{P \in \text{Minh}(R_Q)} e_{\text{HK}}(IR_Q/PR_Q) \lambda(R_P).$$

If  $P \in \text{Minh}(R_Q)$ , by definition,  $\dim R_Q/PR_Q = \text{ht } Q$ . So, since  $Q \in \text{Minh}(I)$  and  $\dim R/I + \text{ht } I = \dim R$ ,  $P \in \text{Minh}(R)$ .

Let  $\Lambda = \cup \text{Minh}(R_Q) \subseteq \text{Minh}(R)$  where the union is taken over all  $Q \in \text{Minh}(I)$ . In the formula above, we change the order of summations to obtain

$$\sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q) = \sum_{P \in \Lambda} \lambda(R_P) \sum_{\substack{Q \in \text{Minh}(I) \\ P \in \text{Minh}(R_Q)}} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q/PR_Q).$$

By the observation in the beginning of the proof,

$$\sum_{\substack{Q \in \text{Minh}(I) \\ P \in \text{Minh}(R_Q)}} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q/PR_Q) = \sum_{Q' \in \text{Minh}(IR/P)} e_{\text{HK}}(J, R/Q') e_{\text{HK}}(IR'_Q/PR'_Q).$$

If the first sum is not empty (i.e., when  $P \subseteq Q$  for some  $Q \in \text{Minh}(I)$ ), then  $J$  is still a system of parameters modulo  $I + P$  because it is a system of parameters modulo  $Q$ . Thus, in this case, by Lemma 5.2,

$$\sum_{\substack{Q \in \text{Minh}(I) \\ P \in \text{Minh}(R_Q)}} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q/PR_Q) \leq e_{\text{HK}}(I + J, R/P).$$

But now, we can use the associativity formula for  $I + J$ , so

$$\begin{aligned} \sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q) &= \sum_{P \in \Lambda} \lambda(R_P) \sum_{\substack{Q \in \text{Minh}(I) \\ P \in \text{Minh}(R_Q)}} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q/PR_Q) \\ &\leq \sum_{P \in \text{Minh}(R)} \lambda(R_P) e_{\text{HK}}(I + J, R/P) = e_{\text{HK}}(I + J), \end{aligned}$$

which finishes the proof.  $\square$

**Corollary 5.24.** *Let  $(R, \mathfrak{m})$  be a local ring of characteristic  $p > 0$  and  $I$  be an ideal such that  $\text{ht } I + \dim R/I = \dim R$ . Let  $J$  be a system of parameters modulo  $I$ . Then*

$$e_{\text{HK}}(I + J) = \sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q)$$

*if and only if*

$$e_{\text{HK}}(I + J, R_{\text{red}}) = \sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(I(R_{\text{red}})_Q).$$

*Proof.* This follows from Lemma 5.23, since both conditions are equivalent for  $R$  and  $R_{\text{red}}$ .  $\square$

The next lemma shows that equimultiplicity is stable under completion. Before starting the proof, we want to recall that Hilbert-Kunz multiplicity is well-behaved with respect to completion, namely  $e_{\text{HK}}(I, M) = e_{\text{HK}_{\widehat{R}}}(I\widehat{R}, M \otimes_R \widehat{R})$  for any finite  $R$ -module  $M$  and  $\mathfrak{m}$ -primary ideal  $I$  in a local ring  $(R, \mathfrak{m})$ .

**Lemma 5.25.** *Let  $(R, \mathfrak{m})$  be a local ring of positive characteristic  $p > 0$  and  $I$  be an ideal such that  $\text{ht } I + \dim R/I = \dim R$ . Let  $J$  be a system of parameters modulo  $I$ . Then*

$$e_{\text{HK}}(I + J) = \sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q)$$

*if and only if*

$$e_{\text{HK}}((I + J)\widehat{R}) = \sum_{P \in \text{Minh}(I\widehat{R})} e_{\text{HK}}(J\widehat{R}/P, \widehat{R}/P) e_{\text{HK}}(I\widehat{R}_P).$$

*Proof.* First, we observe that  $e_{\text{HK}}(I + J) = e_{\text{HK}}((I + J)\widehat{R})$  so we need to show that the right-hand sides are equal.

Let  $Q \in \text{Minh}(I)$ . Because  $\widehat{R/Q} = \widehat{R}/Q\widehat{R}$ ,  $e_{\text{HK}}(J, R/Q) = e_{\text{HK}}(J\widehat{R}/Q, \widehat{R}/Q)$ . So, using the associativity formula for  $e_{\text{HK}}(J\widehat{R}/Q, \widehat{R}/Q)$ ,

$$e_{\text{HK}}(J, R/Q) = e_{\text{HK}}(J\widehat{R}/Q, \widehat{R}/Q) = \sum_{P \in \text{Minh}(Q\widehat{R})} e_{\text{HK}}(J\widehat{R}/P, \widehat{R}/P) \lambda(\widehat{R}_P/Q\widehat{R}_P).$$

Since there is a flat map  $R_Q \rightarrow \widehat{R}_Q \rightarrow \widehat{R}_P$ , it follows that  $e_{\text{HK}}(IR_Q) \lambda(\widehat{R}_P/Q\widehat{R}_P) = e_{\text{HK}}(I\widehat{R}_P)$ . Therefore

$$\sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q) = \sum_{Q \in \text{Minh}(I)} \sum_{P \in \text{Minh}(Q\widehat{R})} e_{\text{HK}}(J\widehat{R}/P, \widehat{R}/P) e_{\text{HK}}(I\widehat{R}_P).$$

Moreover,  $\cup_Q \text{Minh}(Q\widehat{R}) = \text{Minh}(I\widehat{R})$ , because  $\dim R/I = \dim \widehat{R}/I\widehat{R} = \dim \widehat{R}/Q\widehat{R}$ . Thus we obtain that

$$\sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(J, R/Q) e_{\text{HK}}(IR_Q) = \sum_{P \in \text{Minh}(I\widehat{R})} e_{\text{HK}}(J\widehat{R}/P, \widehat{R}/P) e_{\text{HK}}(I\widehat{R}_P).$$

$\square$

**Corollary 5.26.** *Let  $(R, \mathfrak{m})$  be an excellent equidimensional local ring of characteristic  $p > 0$  and let  $I$  be an ideal. Then the following are equivalent:*

- (1)  *$I$  satisfies colon capturing (as in Definition 5.15),*
- (2) *for some (equivalently, every) ideal  $J$  which is a system of parameters modulo  $I$ ,*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P).$$

*Proof.* By Proposition 2.2 and Corollary 5.24, both conditions are independent of the nilradical. Thus we can assume that  $R$  is reduced, so since  $R$  is excellent, by Theorem 2.4, it has a locally stable test element. Last, since  $R$  is an excellent equidimensional reduced ring, it is formally unmixed and we can apply Theorem 5.17.  $\square$

**Corollary 5.27.** *Let  $(R, \mathfrak{m})$  be a local ring of positive characteristic  $p > 0$  and  $I$  be an ideal such that  $\text{ht } I + \dim R/I = \dim R$ . If for some system of parameters  $J = (x_1, \dots, x_m)$  modulo  $I$ ,*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(I, R_P),$$

*then same is true for all systems of parameters.*

*Proof.* First, we use Lemma 5.25 to reduce the question to the completion of  $R$ , note that  $\text{ht } \widehat{IR} + \dim \widehat{R}/\widehat{IR} = \dim \widehat{R}$ . Thus we assume that  $R$  is complete.

Now, condition (a) of Lemma 5.23 is independent of  $J$ . So, it is enough to show that the claim holds in a complete domain. But a complete domain has a locally stable test element by Theorem 2.4 and the claim follows from Proposition 5.14.  $\square$

**5.4. Localization and specialization.** It is time study what happens to Hilbert-Kunz multiplicity after localization and specialization. We will show that localization works pretty generally and establish a Bertini-type theorem for specialization. As a corollary, we will recover condition (c) of Theorem 1.1.

Localization of Hilbert-Kunz multiplicity is a direct consequence of the colon capturing property.

**Proposition 5.28.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring with a test a locally stable test element  $c$  or an excellent locally equidimensional local ring. Let  $I$  be a Hilbert-Kunz equimultiple ideal and  $Q$  be a prime ideal containing  $I$ . Then  $IR_Q$  is still Hilbert-Kunz equimultiple.*

*Proof.* Let  $x_1, \dots, x_n$  be a part of a system of parameters modulo  $I$  that descends to a system of parameters in  $R_Q$  modulo  $IR_Q$ . By the colon capturing property in  $R$  (Corollary 5.26, Theorem 5.17), for all  $i$   $x_{i+1}$  is not a zerodivisor modulo  $((I, x_1, \dots, x_i)^{[q]})^* R_Q$ . Hence, by Lemma 2.12,  $x_{i+1}$  is a regular element modulo  $((I, x_1, \dots, x_i)^{[q]} R_Q)^*$ . Thus  $IR_Q$  is Hilbert-Kunz equimultiple.  $\square$

Now we turn our attention to specialization. Let us recall a slight generalization of a result of Kunz ([15, Proposition 3.2], [30, Proposition 2.13]).

**Proposition 5.29.** *Let  $(R, \mathfrak{m})$  be a local ring and let  $x$  be an element of a system of parameters. Then  $e_{\text{HK}}(I) \leq e_{\text{HK}}(IR/(x))$ .*

**Proposition 5.30.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring with a test a locally stable test element  $c$ . If  $I$  is a Hilbert-Kunz equimultiple ideal, then there exists  $x \in \mathfrak{m}$ , a parameter element modulo  $I$ , such that for every parameter ideal  $J$  modulo  $(I, x)$*

$$e_{\text{HK}}(I + J, R/(x)) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}((J, x), R/P).$$

*Proof.* If  $\dim R = 1$ , any  $P \in \text{Minh}(I)$  is a minimal prime. Take a parameter element  $x$  of  $R$ . Since  $I$  is Hilbert-Kunz equimultiple

$$e_{\text{HK}}(I, x) = \sum_{Q \in \text{Minh}(I)} e_{\text{HK}}(I, R_Q) e_{\text{HK}}(xR/Q) = \sum_{Q \in \text{Min}(I)} \lambda(R_Q) e_{\text{HK}}(xR/Q),$$

where we have used that  $R_Q$  is Artinian. On the other hand, by the associativity formula

$$e_{\text{HK}}(I, x) = \sum_{P \in \text{Minh}(R)} \lambda(R_P) e_{\text{HK}}((I, x)R/P).$$



If  $P$  contains  $I$ , then  $e_{\text{HK}}((I, x)R/P) = e_{\text{HK}}(x, R/P)$ , so after comparing the two formulas we see that  $\text{Minh}(I) = \text{Minh}(R)$ . Since  $R$  is unmixed,  $x$  is not a zero divisor, so by the associativity formula and Proposition 2.15

$$e_{\text{HK}}(IR/(x)) = \lambda(R/(x)) = e(x) = \sum_{P \in \text{Minh}(R)} \lambda(R_P) e_{\text{HK}}(x, R/P) = e_{\text{HK}}(I, x).$$

Now, suppose that  $\dim R \geq 2$ . As in Corollary 5.26, we may assume that  $R$  has a locally stable test element  $c$ . By prime avoidance, there exists a parameter element  $x$  modulo  $I$  that to any minimal prime of  $(c)$ . Take an arbitrary parameter ideal  $J$  modulo  $(I, x)$ . Since  $I$  satisfies colon capturing,  $x$  is regular modulo  $((I + J)^{[q]})^*$  for all  $q$ . Then, by Proposition 2.15 and the associativity formula

$$\lambda\left(R/(((I + J)^{[q]})^*, x)\right) = e\left(x, R/((I + J)^{[q]})^*\right) = \sum_{Q \in \text{Minh}(I+J)} e_{\text{HK}}(x, R/Q) \lambda\left(R_Q/((I + J)^{[q]})^* R_Q\right).$$

Further consider an exact sequence

$$R \xrightarrow{c} R \rightarrow R/(c) \rightarrow 0$$

and tensor it with  $R/((I + J)^{[q]}, x)$ . Since  $c((I + J)^{[q]})^* \subseteq (I + J)^{[q]}$ , the sequence

$$R/(x, ((I + J)^{[q]})^*) \xrightarrow{c} R/(x, (I + J)^{[q]}) \rightarrow R/(c, x, (I + J)^{[q]}) \rightarrow 0$$

is also exact. Note that  $\dim R/(x, c) = \dim R/(x) - 1$ , so after passing to the limit we obtain that

$$e_{\text{HK}}((I + J)R/(x)) \leq \lim_{q \rightarrow \infty} \frac{\lambda\left(R/(((I + J)^{[q]})^*, x)\right)}{q^{d-1}}$$

Thus the formula above gives us that

$$e_{\text{HK}}((I + J)R/(x)) \leq \sum_{Q \in \text{Minh}(I+J)} e_{\text{HK}}(x, R/Q) e_{\text{HK}}((I + J)R_Q).$$

Proposition 5.29 gives us the opposite inequality, so

$$e_{\text{HK}}((I + J)R/(x)) = \sum_{Q \in \text{Minh}(I+J)} e_{\text{HK}}(x, R/Q) e_{\text{HK}}((I + J)R_Q).$$

This gives the statement if  $\dim R/I = 1$ . Otherwise, we use that  $IR_Q$  is still equimultiple, so

$$e_{\text{HK}}((I + J)R_Q) = \sum_{Q \supseteq P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}(J, R_Q/PR_Q).$$

Plugging this in the previous formula and changing the order of summation we get that

$$\begin{aligned} e_{\text{HK}}((I + J)R/(x)) &= \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) \sum_{Q \in \text{Minh}(JR/P)} e_{\text{HK}}(x, R/Q) e_{\text{HK}}(JR_Q/PR_Q) \\ &= \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}((x, J), R/P), \end{aligned}$$

where the last equality holds by Proposition 2.18.  $\square$

As a first corollary we obtain a Bertini-type theorem for Hilbert-Kunz equimultiplicity.

**Corollary 5.31.** *Let  $(R, \mathfrak{m})$  be a local ring which is either formally unmixed with a test a locally stable test element  $c$  or excellent and locally equidimensional. If  $I$  is a Hilbert-Kunz equimultiple ideal, then there exists  $x \in \mathfrak{m}$ , a parameter element modulo  $I$ , such that  $IR/(x)$  is Hilbert-Kunz equimultiple.*

*Proof.* Choose  $x$  as given by the proposition. Let  $(J, x)$  be a system of parameters modulo  $I$ . By Lemma 5.2 and Proposition 5.29 we always have inequalities

$$e_{\text{HK}}((I + J)R/(x)) \geq e_{\text{HK}}(I + (J, x)) \geq \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}((J, x)R/P)$$

and

$$e_{\text{HK}}((I + J)R/(x)) \geq \sum_{Q \in \text{Minh}(I, x)} e_{\text{HK}}(IR_Q/xR_Q) e_{\text{HK}}(JR/Q).$$

By the associativity formula in Proposition 2.18

$$e_{\text{HK}}((J, x)R/P) = \sum_{Q \in \text{Minh}(P, x)} e_{\text{HK}}(JR/Q) e_{\text{HK}}(xR_Q/PR_Q),$$

so combining this with Lemma 5.2 and changing the order of summation we obtain that

$$e_{\text{HK}}((I + J)R/(x)) \geq \sum_{Q \in \text{Minh}(I, x)} e_{\text{HK}}(IR_Q/xR_Q) e_{\text{HK}}(JR/Q) \geq \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}((J, x)R/P)$$

and the corollary follows.  $\square$

In the most interesting case we obtain the following statement.

**Corollary 5.32.** *Let  $(R, \mathfrak{m})$  be a formally unmixed local ring with a test element  $c$ . Let  $\mathfrak{p}$  be a prime ideal of  $R$  such that  $R/\mathfrak{p}$  is a regular ring. Then the following are equivalent*

- (1)  $e_{\text{HK}}(\mathfrak{p}) = e_{\text{HK}}(R/(x))$  for some minimal generator  $x$  of  $\mathfrak{m}$  modulo  $\mathfrak{p}$ ,
- (2)  $e_{\text{HK}}(\mathfrak{p}) = e_{\text{HK}}(R/(y))$  for some parameter  $y$  of  $\mathfrak{m}$  modulo  $\mathfrak{p}$ ,
- (3)  $e_{\text{HK}}(\mathfrak{p}) = e_{\text{HK}}(\mathfrak{m})$ .

*Proof.* (1)  $\Rightarrow$  (2) is obvious. By Proposition 5.29,  $e_{\text{HK}}(R) \leq e_{\text{HK}}(R/(y))$  for any parameter  $y$  in  $R$ . Thus, we always have inequalities  $e_{\text{HK}}(\mathfrak{p}) \leq e_{\text{HK}}(\mathfrak{m}) \leq e_{\text{HK}}(R/(y))$ , and (2)  $\Rightarrow$  (3) follows.

For the last implication we use Corollary 5.31 and observe that in the proof of Proposition 5.30  $x$  can be taken to be a minimal generator of  $\mathfrak{m}$ : an element of  $\mathfrak{m}$  not contained in  $\mathfrak{m}^2$  and the minimal primes of  $c$ .  $\square$

*Remark 5.33.* In many cases, we should be able to choose any minimal generator of  $\mathfrak{m}$  modulo  $\mathfrak{p}$  in the lemma above. Namely, this will hold if the ideal generated by test elements has height at least two; for example, if  $R$  is an excellent normal domain. In this case we will be able to choose a test element  $c$  such that  $\dim R/(c, x) \leq \dim R - 2$ .

Finally, let us recover a Hilbert-Kunz analogue of condition (c) in Theorem 1.1.

**Corollary 5.34.** *Let  $(R, \mathfrak{m})$  be a local ring which is either formally unmixed with a test a locally stable test element  $c$  or excellent and locally equidimensional. If  $I$  is an ideal then  $I$  is Hilbert-Kunz equimultiple if and only if there is a system of parameter  $J = (x_1, \dots, x_r)$  modulo  $I$  which is a part of a system of parameters in  $R$ , and such that*

- if  $r < \dim R$  then  $\sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(IR_P) = e_{\text{HK}}(IR/J)$
- if  $r = \dim R$  then  $\sum_{P \in \text{Minh}(I)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(IR_P) = e_{\text{HK}}(J)$ ,

*Proof.* First suppose that  $I$  is Hilbert-Kunz equimultiple. If  $r = \dim R$ , then we can use the argument in the first part of the proof of Proposition 5.30 and obtain that  $\text{Minh}(I) = \text{Minh}(R)$ . Thus by the associativity formula for any system of parameters  $J$

$$\sum_{P \in \text{Minh}(R)} e_{\text{HK}}(J, R/P) e_{\text{HK}}(IR_P) = \sum_{P \in \text{Minh}(R)} e_{\text{HK}}(J, R/P) \lambda(R_P) = e_{\text{HK}}(J).$$

In general, we build  $J$  by repeatedly applying Proposition 5.30 in  $R/(x_1, \dots, x_i)$  until  $i = r$ . Namely, suppose that we have found a sequence of elements  $x_1, \dots, x_i$  such that for any parameter ideal  $J_i$  in  $R/(I, x_1, \dots, x_i)$

$$e_{\text{HK}}((I + J_i)R/(x_1, \dots, x_i)) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}((J_i, x_1, \dots, x_i)R/P).$$

By Proposition 5.31 and its proof we can find  $x_{i+1}$  such that for every parameter ideal  $J_{i+1}$  in  $R/(I, x_1, \dots, x_i, x_{i+1})$

$$\begin{aligned} e_{\text{HK}}((I + J')R/(x_1, \dots, x_{i+1})) &= e_{\text{HK}}((I, J', x_{i+1})R/(x_1, \dots, x_i)) \\ &= \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}((J_i, x_1, \dots, x_i)R/P). \end{aligned}$$

For the converse, we note that a repeated use of Proposition 5.29 we yield the inequality

$$e_{\text{HK}}(IR/J) \geq e_{\text{HK}}(I + J) \geq \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}(JR/P).$$

□

## 6. APPLICATIONS

It is time to discuss the applications of the developed theory. First, we will show how equimultiplicity forces tight closure of Frobenius powers to be unmixed. It is important this happens to be an equivalent condition in dimension one, and, furthermore, we can make it global and apply to study Hilbert-Kunz multiplicity on the Brenner-Monsky hypersurface.

**6.1. Equimultiplicity and unmixedness of tight closure.** We start with the following consequence of our machinery.

**Proposition 6.1.** *Let  $(R, \mathfrak{m})$  be a local ring of positive characteristic  $p > 0$ . Suppose that either  $R$  is formally unmixed with a locally stable test element  $c$  or is excellent and equidimensional. If  $I$  is an equimultiple ideal for Hilbert-Kunz multiplicity, that is if*

$$e_{\text{HK}}(I + J) = \sum_{P \in \text{Minh}(I)} e_{\text{HK}}(IR_P) e_{\text{HK}}(JR/P)$$

*for some (all) ideals  $J$  generated by a system of parameters modulo  $I$ , then  $\text{Ass}(I^{[q]})^* = \text{Minh}(I)$  for all  $q$ .*

*Proof.* By Theorem 5.17 and Corollary 5.26, we know that  $I$  satisfies colon capturing.

Any prime ideal  $\mathfrak{q} \supset I$  such that  $\mathfrak{q} \notin \text{Minh}(I)$  will contain a parameter element  $x$ . Then by the colon capturing property

$$(I^{[q]})^* : x = (I^{[q]})^*,$$

so  $\mathfrak{q}$  is not an associated prime. □

**Corollary 6.2.** *Let  $(R, \mathfrak{m})$  be a local ring of positive characteristic  $p > 0$  and further suppose that either  $R$  is formally unmixed with a locally stable test element  $c$  or is excellent and equidimensional.*

*If  $\mathfrak{p}$  is a prime ideal such that  $R/\mathfrak{p}$  is regular and  $e_{\text{HK}}(\mathfrak{p}) = e_{\text{HK}}(\mathfrak{m})$ , then  $(\mathfrak{p}^{[q]})^*$  is  $\mathfrak{p}$ -primary for all  $q$ .*

It is extremely useful that the converse holds if  $R/\mathfrak{p}$  has dimension one (Corollary 5.18). But before we will show this importance, we need to globalize our criterion. We start with the following well-known lemma ([2, Lemma 3.5]).

**Lemma 6.3.** *Let  $R$  be a ring of characteristic  $p > 0$  with a test element  $c$  and  $I$  be an ideal of  $R$ . Then  $I^*S^{-1}R = (IS^{-1}R)^*$  for any multiplicatively closed subset  $S$  disjoint from  $\bigcup_q \text{Ass}(R/(I^{[q]}))^*$ .*

**Lemma 6.4.** *Let  $R$  be a ring of characteristic  $p > 0$ . Suppose  $R$  has a test element  $c$ . The following are equivalent:*

- (a)  $(\mathfrak{p}^{[q]})^*$  are  $\mathfrak{p}$ -primary for all  $q$ ,
- (b)  $(\mathfrak{p}^{[q]}R_{\mathfrak{q}})^*$  are  $\mathfrak{p}$ -primary for all  $q$  and all prime ideals  $\mathfrak{q} \supseteq \mathfrak{p}$ ,
- (c)  $(\mathfrak{p}^{[q]}R_{\mathfrak{m}})^*$  are  $\mathfrak{p}$ -primary for all  $q$  and all maximal ideals  $\mathfrak{m}$  that contain  $\mathfrak{p}$ .

*Proof.* The previous lemma shows (a)  $\Rightarrow$  (b) and (b) clearly implies (c). So, we need to show (c)  $\Rightarrow$  (a).

Suppose there exists an embedded prime ideal  $\mathfrak{q}$ . Then there is  $u$  such that  $\mathfrak{q}u \subseteq (\mathfrak{p}^{[q]})^*$ , but  $u \notin (\mathfrak{p}^{[q]})^*$ . A result of Hochster and Huneke ([10, Proposition 6.1]) asserts that a tightly closed ideal is the intersection of the tightly closed ideals containing  $I$  that are primary to a maximal ideal. Therefore  $\mathfrak{p}^{[q]}$  is contained in an ideal  $J$  primary to some maximal ideal  $\mathfrak{m}$  and such that  $u \notin J^*$ . Now Lemma 6.3 shows that  $(J)^*R_{\mathfrak{m}} = (JR_{\mathfrak{m}})^*$ , so, since  $J^*$  is also  $\mathfrak{m}$ -primary,  $u \notin (JR_{\mathfrak{m}})^*$ .

On the other hand,

$$u\mathfrak{p}'R_{\mathfrak{m}} \subseteq (\mathfrak{p}^{[q]})^*R_{\mathfrak{m}} \subseteq (\mathfrak{p}^{[q]}R_{\mathfrak{m}})^* \subseteq (JR_{\mathfrak{m}})^*.$$

Thus  $u \notin (\mathfrak{p}^{[q]}R_{\mathfrak{m}})^*$  and  $\mathfrak{p}'R_{\mathfrak{m}}$  consists of zero divisors on  $(\mathfrak{p}^{[q]}R_{\mathfrak{m}})^*$ . □

Now, we can give a global version of Corollary 6.2.

**Corollary 6.5.** *Let  $R$  be a locally equidimensional ring and let  $\mathfrak{p}$  be a prime ideal such that  $R/\mathfrak{p}$  is regular. If  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$  for all maximal (equivalently, all prime) ideals  $\mathfrak{m}$  containing  $\mathfrak{p}$ , then  $(\mathfrak{p}^{[q]})^*$  are  $\mathfrak{p}$ -primary for all  $q$ .*

The following result is a global version of the equimultiplicity criterion.

**Corollary 6.6.** *Let  $R$  be an excellent domain and let  $\mathfrak{p}$  be a prime ideal such that  $R/\mathfrak{p}$  is a regular ring of dimension one. Then  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(\mathfrak{p})$  for all maximal ideals  $\mathfrak{m}$  containing  $\mathfrak{p}$  if and only if  $(\mathfrak{p}^{[q]})^*$  is  $\mathfrak{p}$ -primary for all  $q$ .*

*Remark 6.7.* It seems that equimultiplicity is a strong condition.

For simplicity, suppose  $R$  is an excellent domain and  $\mathfrak{p}$  is a prime ideal of dimension 1. Then by the previous corollary, if there exists an open equimultiple subset of  $\text{Max } R/\mathfrak{p}$ , then we can find an element  $f \notin \mathfrak{p}$  such that  $(\mathfrak{p}^{[q]}R_f)^*$  is  $\mathfrak{p}$ -primary for all  $q$ . In view of Lemma 6.3, this forces tight closure of all  $\mathfrak{p}^{[q]}R_f$  to commute with localization at any multiplicatively closed set.

This suggests that equimultiplicity can be used to find further examples where tight closure does not commute with localization. We are also able to use the Brenner-Monsky counter-example to study Hilbert-Kunz multiplicity.

**6.2. The Brenner-Monsky example.** It is time to show that our methods can be used to establish some interesting results in well-known example. First, let us introduce the Brenner-Monsky hypersurface

$$R = F[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2),$$

where  $F$  is an algebraic closure of  $\mathbb{Z}/2\mathbb{Z}$ . This hypersurface parametrizes a family of quadrics studied by Monsky in [22].

Since  $R$  is a quotient of a polynomial ring over an algebraically closed field, it is  $F$ -finite. Also,  $R$  is a domain, so it has a locally stable test element. Let  $P = (x, y, z)$  then  $R/P \cong F[t]$  is a regular ring and  $P$  is prime. In [8], Brenner and Monsky showed that tight closure does not commute with localization at  $P$ . Namely, they showed that  $y^3z^3 \notin (P^{[4]})^*$ , but the image of  $y^3z^3$  is contained in  $(P^{[4]}S^{-1}R)^*$  for  $S = F[t] \setminus \{0\}$ .

We want to understand the values of Hilbert-Kunz multiplicity on the maximal ideals containing  $P$ . First, we will need the following result of Monsky.

**Theorem 6.8.** *Let  $K$  be an algebraically closed field of characteristic 2. For  $\alpha \in K$  let  $R_\alpha = K[[x, y, z]]/(z^4 + xyz^2 + (x^3 + y^3)z + \alpha x^2 y^2)$ . Then*

- (1)  $e_{\text{HK}}(R_\alpha) = 3 + \frac{1}{2}$ , if  $\alpha = 0$ ,
- (2)  $e_{\text{HK}}(R_\alpha) = 3 + 4^{-m}$ , if  $\alpha \neq 0$  is algebraic over  $\mathbb{Z}/2\mathbb{Z}$ , where  $m = [\mathbb{Z}/2\mathbb{Z}(\lambda) : \mathbb{Z}/2\mathbb{Z}]$  for  $\lambda$  such that  $\alpha = \lambda^2 + \lambda$
- (3)  $e_{\text{HK}}(R_\alpha) = 3$  if  $\alpha$  is transcendental over  $\mathbb{Z}/2\mathbb{Z}$ .

*Proof.* The last two cases are computed by Monsky in [22]. For the first case, we note that in characteristic 2 we can factor

$$z^4 + xyz^2 + (x^3 + y^3)z = z(x + y + z)((x + y + z)^2 + zy).$$

Thus by the associativity formula,

$$e_{\text{HK}}(R_0) = e_{\text{HK}}(K[x, y]) + e_{\text{HK}}(K[x, y, z]/(x + y + z)) + e_{\text{HK}}(K[x, y, z]/((x + y + z)^2 + zy))$$

and the claim follows.  $\square$

Using the developed machinery we derive the following result from Monsky's computations.

**Proposition 6.9.** *Let  $R = F[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2 y^2)$ , where  $F$  is the algebraic closure of  $\mathbb{Z}/2\mathbb{Z}$ . Then  $e_{\text{HK}}(P) = 3$  for a prime ideal  $P = (x, y, z)$  in  $R$ , but  $e_{\text{HK}}(\mathfrak{m}) > 3$  for any maximal ideal  $\mathfrak{m}$  containing  $P$ .*

*Proof.* First of all, in the notation of the preceding theorem, Cohen's structure theorem ([19, p.211]) shows that  $\widehat{R_P} \cong R_t$  for  $K = F(t)$ , so  $e_{\text{HK}}(R_P) = e_{\text{HK}}(\widehat{R_P}) = 3$ .

Second, since  $F$  is algebraically closed, all maximal ideals containing  $P$  are of the form  $(P, t - \alpha)$  for  $\alpha \in F$ . By Monsky's result,  $e_{\text{HK}}(R_{\mathfrak{m}}/(t - \alpha)) > 3 = e_{\text{HK}}(P)$ , since  $\alpha$  is algebraic. So, since  $R/(t - \alpha)$  is reduced,  $e_{\text{HK}}(\mathfrak{m}) > e_{\text{HK}}(P)$  by Corollary 5.32.  $\square$

We list two easy consequences of this result.

**Corollary 6.10.** *The stratum  $\{\mathfrak{p} \mid e_{\text{HK}}(\mathfrak{p}) = 3\}$  is not locally closed.*

*In particular, the set  $\{\mathfrak{q} \mid e_{\text{HK}}(\mathfrak{q}) \leq 3\}$  is not open.*

*Proof.* If it was an intersection of a closed set  $V$  and open set  $U$ , then the intersection  $V(P) \cap U$  should be non-empty and open in  $V(P)$ . In particular, it would contain infinitely many maximal ideals  $\mathfrak{m}$  containing  $P$ .  $\square$

**Corollary 6.11.** *The set  $\{e_{\text{HK}}(\mathfrak{m}) \mid P \subset \mathfrak{m}\}$  is infinite.*

*Proof.* In the notation of the proof of Proposition 6.9, we have that  $e_{\text{HK}}(\mathfrak{m}R/(t - \alpha)) \geq e_{\text{HK}}(\mathfrak{m}) > e_{\text{HK}}(P)$ . Now the claim follows from Theorem 6.8, since  $e_{\text{HK}}(\mathfrak{m})$  tends to 3 when  $[\mathbb{Z}/2\mathbb{Z}(\lambda) : \mathbb{Z}/2\mathbb{Z}]$  grows.  $\square$

Another application of our methods is a quick calculation of the associated primes of  $P^{[q]}$ . Using the calculations that Monsky made to obtain Theorem 6.8, Dinh ([9]) proved that  $\bigcup_q \text{Ass}(P^{[q]})$  is infinite. However, he was only able to show that the maximal ideals corresponding to the irreducible factors of  $1 + t + t^2 + \dots + t^q$  appear as associated primes, while our methods give all associated primes of the Frobenius powers and their tight closures.

**Proposition 6.12.** *In the Brenner-Monsky example,*

$$\bigcup_q \text{Ass}(P^{[q]})^* = \bigcup_q \text{Ass}(P^{[q]}) = \text{Spec}(R/P).$$

*In particular, the set is infinite.*

*Proof.* Clearly,  $P$  is an associated prime, so we only need to check the maximal ideals.

First, we prove that any prime  $\mathfrak{m}$  that contains  $P$  is associated to some  $(P^{[q]})^*$ . If not, then  $(P^{[q]})^*R_{\mathfrak{m}}$  are  $P$ -primary for all  $q$ . Note that  $(P^{[q]})^*R_{\mathfrak{m}} \subseteq (P^{[q]}R_{\mathfrak{m}})^*$ , thus by Corollary 2.13  $(P^{[q]}R_{\mathfrak{m}})^*$  is  $P$ -primary for any  $q$ . Therefore, by Corollary 5.18,  $e_{\text{HK}}(\mathfrak{m}) = e_{\text{HK}}(P)$ , a contradiction.

For the second claim, let  $\mathfrak{m}$  be any maximal ideal containing  $P$ . Since  $e_{\text{HK}}(P) < e_{\text{HK}}(\mathfrak{m})$  and  $R/P$  is regular,  $\mathfrak{m}$  is an associated prime of  $(P^{[q]}R_{\mathfrak{m}})^*$  for some  $q$ . Thus, there exists  $u \notin (P^{[q]}R_{\mathfrak{m}})^*$  such that

$$cu^{q'}\mathfrak{m}^{[q']} \subseteq P^{[qq']}R_{\mathfrak{m}}.$$

Now, if  $\mathfrak{m}$  is not an associated prime of any  $P^{[qq']}$ , then we would have  $u \in (P^{[q]}R_{\mathfrak{m}})^*$ , a contradiction.  $\square$

*Remark 6.13.* The presented example shows that Hilbert-Kunz multiplicity need not be locally constant when tight closure does not commute with localization. However, it is not clear whether it should be locally constant if, in a particular example, tight closure commutes with localization. Even in this case  $\bigcup_q \text{Ass}(\mathfrak{p}^{[q]})^*$  might be infinite ([26]), and it is not clear why the intersection of the embedded primes must be greater than  $\mathfrak{p}$ .

## 7. OPEN PROBLEMS

There is a number of question laying further in the direction and spirit of this work. Here is a small selection of the favorites.

*Question 7.1.* What can be said about the maximum value locus of the Hilbert-Kunz multiplicity? Are the irreducible components regular?

In the Brenner-Monsky example the locus consists of finitely many maximal ideals, so the irreducible components are indeed smooth. The author does not know any computation of Hilbert-Kunz multiplicity in a two-parameter family, so there is no real example to test this question on.

*Question 7.2.* What is the behavior of Hilbert-Kunz multiplicity under a blow-up? If the previous question has a positive answer, we may hope that one can always choose a component such that its blow-up will not increase the Hilbert-Kunz multiplicity.

*Question 7.3.* This work suggests that Hilbert-Kunz multiplicity might be “too sensitive”. Is there a suitable coarsening?

Perhaps, Hilbert-Kunz multiplicity should be paired with a less sensitive invariant, such as the Hilbert-Samuel function, and should be only used if that invariant does not provide much information. Another possible hope is to “round” Hilbert-Kunz multiplicity, perhaps the fact that Hilbert-Kunz multiplicity cannot be too close to 1 is a sign?

It is widely believed that Hilbert-Kunz multiplicity can be defined in characteristic zero by taking the limit of the Hilbert-Kunz multiplicities of the reductions mod  $p$  (see [7, 28] for partial results).

*Question 7.4.* What are the geometric properties of the characteristic zero Hilbert-Kunz multiplicity? Can it be used to give a new proof of Hironaka’s theorem on the resolution of singularities?

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